Counting Nodal Components of Boundary-adapted Arithmetic Random Waves

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Declaration

I, James Cann, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

The 'nodal sets' (zero sets) of Dirichlet Laplace eigenfunctions for the two dimensional unit square have historically raised many questions, and continue to do so today. Prominent amongst them is the question of the number of 'nodal components' (connected components of the zero set) of a typical eigenfunction.

In this thesis, we attribute Gaussian random coefficients to a standard basis of eigenfunctions for each eigenspace, to form the ensemble of 'boundary-adapted arithmetic random waves'. We then study the number of nodal components – now a random variable – of this ensemble as the eigenvalue grows to infinity, and establish the existence of a limiting mean nodal intensity which is non-universal, in the sense that it depends (indeed *relies*) upon restriction to subsequences of eigenvalues with specific arithmetic properties. We further show that the number of nodal components concentrates exponentially in probability about this limiting mean intensity.

Impact Statement

Nodal figures – the sets which remain stationary when a membrane is undergoing vibration – have long been of interest to mathematicians and importance to the wider scientific community, with an understanding of their geometric properties finding application in musical instrument design, oceanography and earthquake science.

But it was the quest to detect chaotic phenomena in quantum mechanical systems that has re-initiated exciting research in the subject. In 1977, Michael Berry made a profound conjecture relating high-energy nodal figures with random ensembles of monochromatic Laplace eigenfunctions. Empirical studies by physicists have lent considerable evidence to Berry's conjecture, and there has arisen a new need to gain a precise understanding of these random ensembles and the geometric properties of their 'random nodal figures'. Recent decades have seen a move towards converting and refining the empirical studies in the physics literature into rigorous mathematical questions, and contemporaneous with this thesis are substantive research programmes with precisely that aim, including the 5-year European Research Council grant 'Nodal Lines'.

The contribution of this thesis to the wider research programme, is to ascertain whether the imposition of boundary conditions in a model of random Laplace eigenfunctions affects the statistics of the nodal count. Our findings raise further questions, in particular, at what level the high-energy statistics of the nodal count discriminate between models with boundary and those without.

The research conducted in this thesis has been delivered to the Houses of Parliament at the 'STEM for Britain 2018' poster event, and presented at the 'Global Young Scientists Summit 2018' in Singapore. Outputs from the work have been disseminated through public engagement, providing cover art for Issue 07 of popular mathematics magazine 'Chalkdust', and featuring in the UCL 'Research Images as Art' 2018 exhibition.

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CHAPTER 1

Introduction

In this chapter we provide historical and mathematical context for the questions addressed in this thesis. We define the ensemble of boundary-adapted arithmetic random waves, state our main results, and close the chapter by outlining the structure of the remainder of the thesis.

1. Laplace eigenfunctions and their nodal sets

The eigenvalue problem for the Laplace operator has a rich history. The search for solutions, together with a description of the associated eigenfunctions, originally arose from the study of vibrating plates and membranes, as mathematicians sought to explain the physical experiments of Robert Hooke and the sound patterns of Ernst Chladni. To the present day, and the description of these eigenfunctions – both analytic and numerical – is of exceptional importance across pure and applied mathematics, with closely related problems pervading physics, engineering and the broader physical sciences.

1.1. Laplace eigenfunctions for a general domain. Fix $d \geq 2$ and suppose $X \subset \mathbb{R}^d$ is a domain, with $\overline{X}$ compact. We begin by considering the problem of describing the real-valued functions $f \in L^2(X)$ which are eigenfunctions of the Laplace operator. If the boundary ∂X of X is non-empty, then we assume the Dirichlet boundary condition $f|_{\partial X} = 0$. Thus we seek to solve

$$\Delta f + Ef = 0,$$

where

$$\Delta = \tfrac{\partial^2}{\partial x_1^2} + \tfrac{\partial^2}{\partial x_2^2} + \ldots + \tfrac{\partial^2}{\partial x_d^2},$$

for some $E \in \mathbb{R}$. Specifying these eigenvalues E for which eigenfunctions exist is part of the challenge.

Solutions to this problem are known to exist for a wide generality of domains, but for those X described above one can prove that the set of eigenvalues (subsuming

multiplicities) consists of a sequence

$$0 < \overline{\gamma}_1 < \overline{\gamma}_2 < \ldots \uparrow +\infty$$

where each associated eigenspace is finite-dimensional, and the eigenfunctions in each eigenspace are C^∞ on $\overline{X}$ [**8**, Theorem 1]. Despite this existence theorem, it is often exceptionally difficult or impossible to actually specify the eigenvalues and their eigenfunctions. Nevertheless, there do exist special domains for which it is possible both to list the eigenvalues and to describe all of the associated eigenfunctions analytically. One such domain – the unit square in $\mathbb{R}^2$ – is the focus of this thesis.

1.2. Dirichlet Laplace eigenfunctions for the square. For the square $\mathbb{S} := [0,1]^2 \subset \mathbb{R}^2$, we can write down each of its Dirichlet Laplace eigenvalues together with a basis of eigenfunctions. To see this, it suffices that a complete orthonormal basis for the Hilbert space $L^2(\mathbb{S})$ is given by the trigonometric functions

$$\phi_\lambda(x) := 2\sin(\lambda_1 \pi x_1)\sin(\lambda_2 \pi x_2) \tag{1.1}$$

where $\lambda = (\lambda_1, \lambda_2)$ runs through $\mathbb{N}^2$. We see immediately that the eigenvalues form a discrete set parametrised by pairs of natural numbers, and by computation on this basis we observe that

$$\Delta\phi_\lambda = -\pi^2(\lambda_1^2 + \lambda_2^2)\phi_\lambda.$$

The spectrum therefore consists of values $E_n = \pi^2 n$ for $n \in S$ where S denotes the set of integers expressible as a sum of two positive integer squares:

$$S = \left\{ n = \lambda_1^2 + \lambda_2^2 : \lambda_1, \lambda_2 \in \mathbb{N} \right\}.$$

There may be many such ways of expressing a given integer $n \in S$, and so each eigenvalue $E_n = \pi^2 n$ comes with a multiplicity which indicates the dimension of the E_n-eigenspace. The properties of the finite set

$$\Lambda_n := \{\lambda \in \mathbb{Z}^2 : \|\lambda\|^2 = n\} \tag{1.2}$$

will be important in this thesis, but perhaps more important still is its subset

$$\Lambda_n^{\mathbb{S}} := \{\lambda \in \mathbb{N}^2 : \|\lambda\|^2 = n\}, \tag{1.3}$$

which we call the *frequency set* corresponding to the *energy level* E_n. The frequency set parametrises a basis of eigenfunctions for the E_n-eigenspace, and the latter we denote as

$$\mathcal{H}_n = \mathrm{Span}_{\mathbb{R}} \left\{ \phi_\lambda : \lambda \in \Lambda_n^{\mathbb{S}} \right\}.$$

The *dimension* of $\mathcal{H}_n$ is

$$\mathcal{N}_n := \dim \mathcal{H}_n = \begin{cases} \frac{1}{4}r_2(n) - 1, & n \in S, n = m^2, \\ \frac{1}{4}r_2(n), & \text{else.} \end{cases}$$

The arithmetic function $r_2(n) := \#\Lambda_n$ is of great significance in number theory, and there are major open problems regarding its behaviour, which is rather erratic as exhibited by:

$$\liminf_{n\to\infty} r_2(n) = 4 \tag{1.4}$$

$$\limsup_{n\to\infty} r_2(n) = \infty. \tag{1.5}$$

Averaged over the set $\{n \le N\} \cap S$, the value of $r_2(n)$ grows on the order of $\sqrt{\log N}$ for large N, which is to say that a typical value of $\mathcal{N}_n$ grows to infinity with n. We will take a closer look at the properties of Λ_n in Section 3.

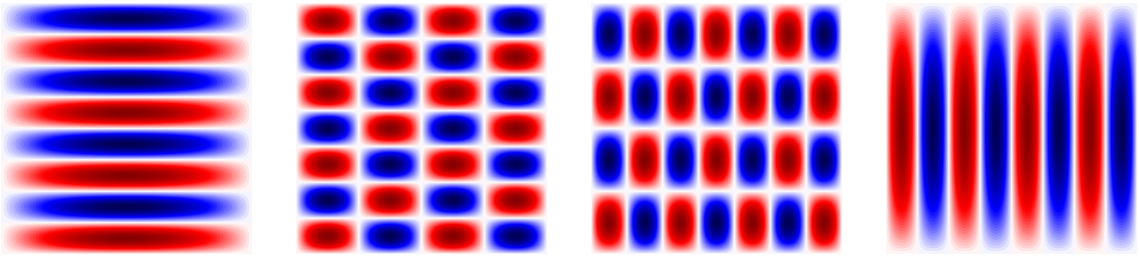

FIGURE 1. A standard basis for the eigenspace $\mathcal{H}_{65}$: the functions $\phi_\lambda(x)$ for $\lambda = (1,8), (4,7), (7,4), (8,1)$, (left to right) shown as contour plots coloured red > 0, white $= 0$, blue < 0.

1.3. Nodal sets of Laplace eigenfunctions. The *nodal set* of *any* function $f : X \to \mathbb{R}$ is defined as

$$Z(f) := \{x \in X : f(x) = 0\} = f^{-1}(0). \tag{1.6}$$

If we suppose that $X \subset \mathbb{R}^2$ is a connected and bounded domain with a sufficiently regular boundary, then the nodal sets of Laplace eigenfunctions on X consist of a finite union of smooth curves, which partition X into disjoint *nodal domains*: the connected components of $X \setminus Z(f)$. It is quite possible that two or more of the smooth nodal curves intersect; if so we treat them as constituting a single *nodal component*. We write $\mathcal{Z}(f)$ for the *set of nodal components* of $Z(f)$, and it is the

number of nodal components

$$N(f) := \#\mathcal{Z}(f),$$

that we will study for the special domain $X = \mathbb{S}$ throughout this thesis.

The nodal sets of Dirichlet Laplace eigenfunctions are worthy of note for both local and global reasons. Globally, there is a celebrated theorem of Courant [**11**] which states that, for any connected and bounded domain X with sufficiently regular boundary, if we list (possibly with repetition) the eigenvalues of the Dirichlet Laplacian in non-decreasing order:

$$0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \ldots \uparrow \infty$$

and specify an eigenfunction f_{γ_n} for each, then the *number of nodal domains* N_d of f_{γ_n} cannot exceed n. In particular, our definition of nodal components implies that

$$N(f_{\gamma_n}) \leq N_d(f_{\gamma_n}) \leq n. \tag{1.7}$$

A beautiful note of Pleijel [**20**] went further, asymptotically sharpening Courant's nodal domain theorem to give

$$\limsup_{n\to\infty} \frac{N_d(f_{\gamma_n})}{n} \leq \left(\frac{2}{j_{0,1}}\right)^2 = 0.691\ldots \tag{1.8}$$

where $j_{0,1}$ is the first zero of the Bessel function J_0. Pleijel gave a neat corollary of this: that N_d can only equal n for a finite number of eigenvalues. He devoted a section of the paper to the square domain $\mathbb{S}$, demonstrating that these so-called *Courant-sharp* eigenvalues are $\gamma_1 = 2$, $\gamma_2 = 4$, and $\gamma_4 = 8$. Stern [**24**] embraced the lower bounds, showing that $N_d = 2$ for an infinite family (f_n) of eigenfunctions for $\mathbb{S}$:

$$\liminf_{n\to\infty} N_d(f_n) = 2. \tag{1.9}$$

Together, the results of Pleijel and Stern leave open a wide variety of possible behaviours for the nodal count $N(f_n)$ for the domain $\mathbb{S}$, superficially analogous to the behaviour of $r_2(n)$. As mentioned in the previous section, the eigenspaces for $\mathbb{S}$ are multidimensional, and so it is natural to wonder what the value of $N(f)$ is for a "typical" eigenfunction f of eigenvalue γ_n.

2. Random ensembles of Laplace eigenfunctions

One way of endowing the notion of "typical" with the requisite mathematical rigour is to consider how $N(f)$ behaves for random linear combinations of a standard

basis of eigenfunctions. The count $N(f)$ becomes a random variable, and one might hope to understand its expectation, higher moments and distribution. Where there is eigenspace degeneracy, such as for $\mathbb{S}$, one can form these random linear combinations for each eigenspace individually. In this section we visit two pertinent examples of Gaussian random ensembles whose study pre-dates this thesis, and then introduce the ensemble of boundary-adapted arithmetic random waves, whose nodal count is the subject of this thesis.

2.1. Random spherical harmonics (RSHs). Spherical harmonics are special functions defined on the two-dimensional unit sphere $\mathcal{S}^2$, which find common utility across mathematics and physics. When expressed in Cartesian coordinates the spherical harmonics of degree n are the restrictions to $\mathcal{S}^2$ of the harmonic and homogeneous polynomials of degree n. They form a $(2n+1)$-dimensional real Hilbert space $\mathcal{L}_n$ when equipped with the $L^2(\mathcal{S}^2)$ norm. The notion of nodal components here is in precise analogy with (1.6): for $f \in \mathcal{L}_n$ we consider

$$Z(f) = \{x \in \mathcal{S}^2 : f(x) = 0\},$$

and let $N(f)$ be the number of connected components of $Z(f)$.

In the ground-breaking paper [**17**], Nazarov and Sodin set out to verify predictions made in a percolation-like lattice model of nodal domains, conceived of by the physicists Bogomolny and Schmit [**3**]. They defined the *random spherical harmonic*

$$f_n(x) = \frac{1}{\sqrt{2n+1}} \sum_{k=-n}^{n} \xi_k Y_k(x) \tag{1.10}$$

where the ξ_k are independent identically distributed standard Gaussian random variables and $\{Y_k\}_{k=-n}^{n}$ is an orthonormal basis of $\mathcal{L}_n$. The nodal count $N(f_n)$ for the random function f_n is a random variable and they studied its distribution. The aforementioned Bogomolny–Schmit model predicted that $\mathbb{E}\{N(f_n)/n^2\}$ tends to a positive limit as $n \to \infty$, and Nazarov and Sodin were not only able to verify this prediction, but showed more:

THEOREM (Exponential concentration for RSH [**17**]). *There exists a positive constant α such that, for all $\epsilon > 0$*

$$\mathbb{P}\left\{\left|\frac{N(f_n)}{n^2} - \alpha\right| > \epsilon\right\} \leq C(\epsilon) e^{-c(\epsilon)n}$$

where $c(\epsilon)$, $C(\epsilon)$ are positive constants depending only on ϵ.

This theorem paved the way for analogous studies of other Gaussian random ensembles, of which the study undertaken in this thesis is one.

2.2. Arithmetic random waves (ARWs). The direct historical predecessor of the main object of study in this thesis is the *ensemble of arithmetic random waves*, introduced by Oravecz, Rudnick and Wigman in [**19**]. For fixed $d \geq 2$, let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the flat d-dimensional torus and consider Laplace eigenfunctions $f \in L^2(\mathbb{T}^d)$. Let $\mathcal{E}_E \subset L^2(\mathbb{T}^d)$ be the real Hilbert space of Laplace eigenfunctions which satisfy

$$\Delta f + 4\pi^2 E f = 0,$$

for some $E \neq 0$. Here, the energy spectrum consists of eigenvalues $4\pi^2 n$, parametrised by those integers n which are representable as the sum of d integer squares; the spectral multiplicity is given by the arithmetic function $r_d(n)$. An L^2-orthogonal basis for $\mathcal{E}_n$ is given by the set $\{e^{2\pi i\langle \lambda, x\rangle}\}_{\lambda\in\Lambda_n}$, where the frequency set corresponding to $E = n$ is now

$$\Lambda_n = \left\{\lambda \in \mathbb{Z}^d : \|\lambda\|^2 = n\right\}.$$

The *arithmetic random wave* $f_n : \mathbb{T}^d \to \mathbb{R}$ was then defined by the authors as

$$f_n(x) = \sqrt{\frac{2}{\dim \mathcal{E}_n}} \sum_{\lambda\in\Lambda_n^+} \left(b_\lambda \cos(2\pi\langle\lambda, x\rangle) + c_\lambda \sin(2\pi\langle\lambda, x\rangle)\right) \tag{1.11}$$

where $\Lambda_n^+ := \Lambda_n/\pm$, and where $b_\lambda, c_\lambda \sim \mathcal{N}(0,1)$ are real i.i.d. Gaussian random variables.

Rudnick and Wigman [**22**] took interest in the high-energy properties of the *nodal volume*: the random variable

$$\mathcal{L}_n := \mathcal{L}(f_n) = \text{length}(f_n^{-1}(0)),$$

computing the expected nodal volume of arithmetic random waves, and conjecturing an upper bound for the variance of $\mathcal{L}_n$. For $d = 2$, Krishnapur, Kurlberg and Wigman [**14**] improved on this by establishing the asymptotic behaviour of the variance of $\mathcal{L}_n$, and demonstrating that the asymptotics depend upon the angular distribution of Λ_n on the circle. Their work sparked a significant amount of interest in the model, and there have been many further studies of the distribution of the nodal volume, as well as nodal intersections with fixed smooth reference curves.

Rozenshein [**21**] considered the nodal count $N(f_n)$ for arithmetic random waves, employing machinery developed in [**18**] to adapt the method in [**17**] for this model.

Rozenshein also proved a very useful result on the measurability of the nodal count in a general setting. The stationarity of the ensemble of arithmetic random waves means that the work of [**23**] quickly establishes existence of a limiting mean nodal intensity, and the main result of [**21**] was to show that exponential concentration holds in this setting too:

THEOREM (Exponential concentration for ARWs [**21**]). *Let $\epsilon > 0$. There exist constants c, $C > 0$ depending only on ϵ, such that, if $\mathbb{E}\left\{N(f_{n_j})/n_j^{d/2}\right\}$ tends to a limit α through some sequence $(n_j)_{j\in\mathbb{N}}$, then for sufficiently large j,*

$$\mathbb{P}\left\{\left|\frac{N(f_{n_j})}{n_j^{d/2}} - \alpha\right| > \epsilon\right\} \leq C(\epsilon)e^{-c(\epsilon)\dim \mathcal{E}_{n_j}}.$$

2.3. Boundary-adapted arithmetic random waves (BARWs). The RSH and ARW ensembles share common features: both of their domains of definition are without boundary, and both are stationary as random fields. Likewise for each, the theorems obtained for concentration about their first moments share a high degree of similarity. From the perspective of random ensembles of Laplace eigenfunctions, one might reasonably raise the question: what happens if one considers an ensemble with eigenspace degeneracy, but also a non-empty boundary with Dirichlet boundary condition; or alternatively, in historic continuation with the studies of Pleijel and Stern, one might seek to understand the "typical" behaviour of the nodal count $N(f)$ for the square domain $\mathbb{S}$ (cf. Section 1.3).

To make this problem rigorous, we use the notation of Section 1.2 to define the ensemble of *boundary-adapted arithmetic random waves* (BARWs) as the collection of Gaussian random fields $f_n : \mathbb{S} \to \mathbb{R}$ consisting of real-valued $\mathcal{H}_n$-functions of eigenvalue $E_n = \pi^2 n$, where

$$f_n(x) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n^{\mathbb{S}}} \xi_\lambda \, \phi_\lambda(x), \tag{1.12}$$

and where the coefficients $\xi_\lambda \sim \mathcal{N}(0,1)$ are real i.i.d. Gaussian random variables. We will assume w.l.o.g. that for at least one $\lambda \in \Lambda_n^{\mathbb{S}}$ we have $\xi_\lambda \neq 0$, which is true almost surely thus not affecting our study of f_n as a random field, and without which the sample f_n is not an E_n-eigenfunction. The normalisation factor $1/\sqrt{\mathcal{N}_n}$ has been chosen so that the point-wise variance $\mathbb{E}\{f_n(x)^2\}$, when averaged over all $x \in \mathbb{S}$, equals 1. That is

$$\int_{\mathbb{S}} \mathbb{E}\left\{f_n(x)^2\right\} dx = 1.$$

This normalisation is arbitrary and does not alter the nodal set in any way, but will simplify some of the calculations which appear later. We think of the boundary-adapted arithmetic random wave f_n as a random element of the finite-dimensional inner product space $\mathcal{H}_n$, for which the number of its nodal components $N(f_n)$ is a random variable[1].

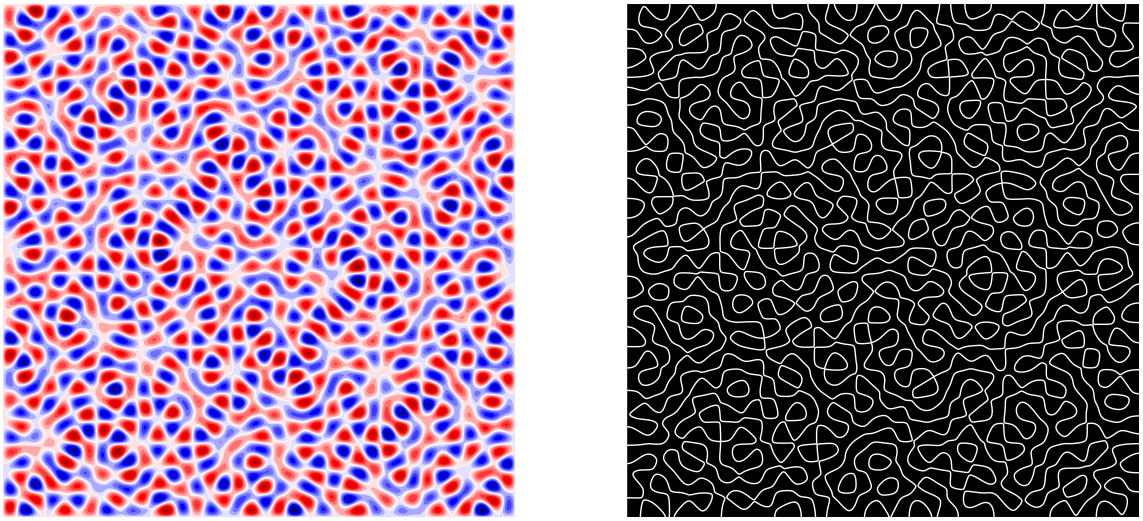

FIGURE 2. A sample of the BARW f_{1885}: its contour plot coloured red $>$ 0, white $=$ 0, blue $<$ 0 (left); its nodal set in white (right).

3. Limiting measures and lattice points on circles

One reason why the study of the nodal lines of boundary-adapted arithmetic random waves is so rich, is because of the arithmetic nature of its frequency sets $\Lambda_n^{\mathbb{S}}$ and their supersets Λ_n. Each $\lambda \in \Lambda_n$ may be thought of as a $\mathbb{Z}^2$ lattice point lying on the circle of radius $\sqrt{n}$, and before stating the main results of this thesis we will need to know something of the asymptotic distribution of these points as the radius of the circle grows to infinity.

A neat way of describing the distribution of lattice points on circles is via probability measures – a perspective which was used to great effect in the work of Krishnapur, Kurlberg and Wigman [**14**] and which we draw upon for the following description. For each $n \in S$, the set Λ_n defines a discrete probability measure μ_n on

[1]See Chapter 2.3 for further details.

the unit circle $\mathcal{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ by setting

$$\mu_n := \frac{1}{r_2(n)} \sum_{\lambda \in \Lambda_n} \delta_{\lambda/\sqrt{n}},$$

where δ_x is the Dirac delta measure supported at x. The set Λ_n has a number of symmetries: it is invariant under the maps $z \mapsto \overline{z}$ and $z \mapsto iz$, and therefore so is the measure μ_n.

Particularly pleasing is the fact that, using properties of the Gaussian integers, we can construct sequences $(n_j)_{j\in\mathbb{N}}$ for which the corresponding sequence of the measures μ_{n_j} tends weakly to the uniform measure supported on the union of four equal length arcs positioned symmetrically around the circle. More precisely, for any $t \in [0, \pi/4]$, define a probability measure ν_t on $\mathcal{S}^1$ by

$$\nu_t(f) := \frac{1}{8t} \sum_{k=0}^{3} \int_{-t+k\frac{\pi}{2}}^{t+k\frac{\pi}{2}} f(e^{i\theta}) d\theta,$$

with the convention that $\nu_0 := \frac{1}{4}\sum_{k=0}^{3} \delta_{i^k}$. It was shown in [**14**] that for every $t \in [0, \pi/4]$ there exists a sequence $(n_j)_{j\in\mathbb{N}}$ such that the counting measure μ_{n_j} converges weakly to the measure ν_t as $j \to \infty$.

There are also atomic limiting measures: distinguished amongst them are the only two limiting measures which are supported on precisely four points: these are ν_0 (as defined above) and $\tilde{\nu_0}$ (the rotation of ν_0 by $\pi/4$), which have come to be known as the *Cilleruelo* and *tilted Cilleruelo* measures, in recognition of the results in [**10**].

There are many more measures ν than those example given above, which are attainable as weak limits of the set $\{\mu_n\}_{n\in S}$. In fact, the existence of ν_0 may be used to show that: if ν is attainable as such a weak limit, then there is a sequence $\{n_j\}_{j\in\mathbb{N}}$ for which μ_{n_j} tends weakly to ν with the additional property that $r_2(n_j) \to \infty$. Kurlberg and Wigman further investigated the structure of the set of attainable measures in [**15**].

4. Main results

So what is the "typical" behaviour of the nodal count for the ensemble of boundary-adapted arithmetic random waves? Our first theorem demonstrates the existence of a non-universal limiting mean nodal intensity:

THEOREM 1.1 (Limiting mean nodal intensity). *Let (f_n) be the ensemble of boundary-adapted arithmetic random waves, and suppose that $(n_j)_{j\in\mathbb{N}}$ is any sequence of natural numbers such that the probability measure μ_{n_j} on $\mathcal{S}^1$ induced by*

Λ_{n_j} converges weakly to a non-atomic limiting measure ν. Then there exists a constant $a_\nu > 0$ such that

$$\mathbb{E}\left\{\left|\frac{N(f_{n_j})}{n_j} - a_\nu\right|\right\} \to 0 \qquad (j \to \infty).$$

REMARK. The constants a_ν were studied by Kurlberg and Wigman [**16**], and are known as *Nazarov–Sodin constants*. Here, a_ν is the Nazarov–Sodin constant of the (spectral) measure ν on the unit circle. Kurlberg and Wigman showed that the only such measures arising from lattice points on circles for which the Nazarov–Sodin constant equals zero, are the Cilleruelo and tilted Cilleruelo measures encountered in Section 3, both of which are atomic. This provides important additional information, implying that: if the constant a_ν for the *non-atomic* measure ν in the statement of Theorem 1.1 exists, then it must be positive.

Somewhat remarkably: when the sequence $(n_j)_{j\in\mathbb{N}}$ is taken such that μ_{n_j} tends weakly to $\frac{d\theta}{2\pi}$ (the normalised Lebesgue measure on the unit circle), the Nazarov–Sodin constant for the ensemble of BARWs and (two-dimensional) ARWs is precisely equal to the limiting constant α for the ensemble of RSHs. However, at the time of writing, it is still not known what the value of this constant is.

With Theorem 1.1 established, one could then ask how likely it is that the random variable $N(f_n)/n$ takes a value near to this limiting mean a_ν. We demonstrate the exact analogy of the exponential concentration results proved previously for the models of ARWs and RSHs:

THEOREM 1.2 (Exponential concentration). *With the assumptions of Theorem 1.1 and with the same constant a_ν appearing there, for all $\epsilon > 0$ there exist positive constants $c(\epsilon), C(\epsilon)$ such that*

$$\mathbb{P}\left\{\left|\frac{N(f_{n_j})}{n_j} - a_\nu\right| > \epsilon\right\} \leq C(\epsilon)e^{-c(\epsilon)\dim \mathcal{H}_{n_j}} \qquad (j \to \infty).$$

REMARK. One might hope for a stronger statement of each of these theorems, where we do not distinguish subsequences $(n_j)_{j\in\mathbb{N}}$ attaining a particular limiting measure; that is we might hope for a statement of the form

$$\mathbb{E}\left\{\left|\frac{N(f_n)}{n} - a_{\mu_n}\right|\right\} \to 0 \qquad (n \to \infty). \tag{1.13}$$

However this cannot hold: Kurlberg and Wigman [**16**] meet this question and find that, while we do have uniform convergence to zero of the sequence of differences $\mathbb{E}\,[N(f_n)/n] - a_{\mu_n}$, the associated *discrepancy* $\mathbb{E}\,|N(f_n)/n - a_{\mu_n}|$ does *not* convergence uniformly to zero as $n \to \infty$.

5. Outline of thesis

The remainder of this thesis is divided into three chapters. Chapter 2 consists for the most part of exposition of the work of others, covering methods and inequalities which will be put to use in later chapters. We flesh out our earlier discussion of Laplace eigenfunctions, we consider BARWs from the perspective of Gaussian random fields, and also discuss the nodal counting methodology of Nazarov and Sodin [**17, 18**] upon which much of this thesis is based. Chapter 3 begins by giving a more technical indication of how BARWs differ from previously studied ensembles of random Laplace eigenfunctions, and how they do not satisfy the hypotheses required to deploy Nazarov and Sodin's methods directly. We demonstrate that BARWs do, however, satisfy a technical adaptation of their hypotheses, and show that the local limiting nodal intensity lemma from [**18**] may be adapted to accommodate these alternative hypotheses. Chapter 4 contains the proofs of the main results of this thesis.

Notation

We will introduce special notation as we progress, but here we clarify some notation which will be used throughout. We write:

- $f \ll g$, if we have $|f(n)| \leq c\,|g(n)|$ for some $c > 0$ as $n \to \infty$;
- $f = O(g)$, as equivalent notation for $f \ll g$;
- $f \sim g$, for positive $f(n)$, $g(n)$, if $f(n)/g(n) \to 1$ as $n \to \infty$;
- $f \asymp g$, if $g \ll f \ll g$.
- $\mu_n \xrightarrow{*} \mu$, for weak convergence of the sequences of measures $\{\mu_n\}_{n\in\mathbb{N}}$ to μ.
- $E_{+\rho}$, for the ρ–neighbourhood of the set E.

CHAPTER 2

Foundations

1. Laplace eigenfunctions

As we saw in the introduction, the nodal sets and nodal domains of Laplace eigenfunctions are worthy of special remark, exemplified by the work of Courant, Pleijel and Stern. Here we review some more detailed results about Laplace eigenfunctions, contenting ourselves with those results which will be required later in the thesis. More comprehensive treatments may be found in [**8**] and [**26**].

1.1. Local properties. The first properties which we draw attention to are those which occur locally, namely within neighbourhoods of every point. We begin by stating a lemma which is recited in [**2**], and proved earlier by Cheng in [**9**].

LEMMA 2.1 (Cheng, Berard–Helffer [**2**]). *Let $x \in \mathbb{S} \setminus \partial\mathbb{S}$ and suppose ϕ is a non-zero Laplace eigenfunction. Then*

(i) *The eigenfunction ϕ cannot vanish to infinite order at x.*
(ii) *If ϕ vanishes at x, then the leading part of its Taylor expansion at x is a harmonic homogeneous polynomial.*
(iii) *If the point x is a* critical zero *of ϕ (namely: $\phi(x) = 0 = |\nabla\phi(x)|$), then its nodal set Z_ϕ consists of finitely many regular arcs forming an equiangular system meeting at x.*
(iv) *The nodal set Z_ϕ can only have self-intersections at critical zeroes, and the number of arcs which meet at a self-intersection is determined by the order of vanishing of the eigenfunction. Nodal lines cannot meet tangentially.*
(v) *The nodal set Z_ϕ cannot have an endpoint within $\mathbb{S} \setminus \partial\mathbb{S}$, and consists of finitely many analytic arcs.*
(vi) *If ϕ has eigenvalue λ, then the first Dirichlet eigenvalue of each nodal domain is λ.*
(vii) *Property (iii) is also true for $x \in \partial\mathbb{S}$.*

Most of these properties are observable in Fig. 3, which shows the nodal set of a random sample from the BARW f_{1885} in a blown-up central disc. Lemma 2.1 validates and appends further detail to our passing statement in Section 1.3 that:

The nodal sets of Laplace eigenfunctions f_λ are remarkable and consist of a finite union of smooth curves, which partition $\mathbb{S}$ into disjoint nodal domains. It is quite possible that two or more of the smooth curves intersect; if so we treat them as constituting a single nodal component.

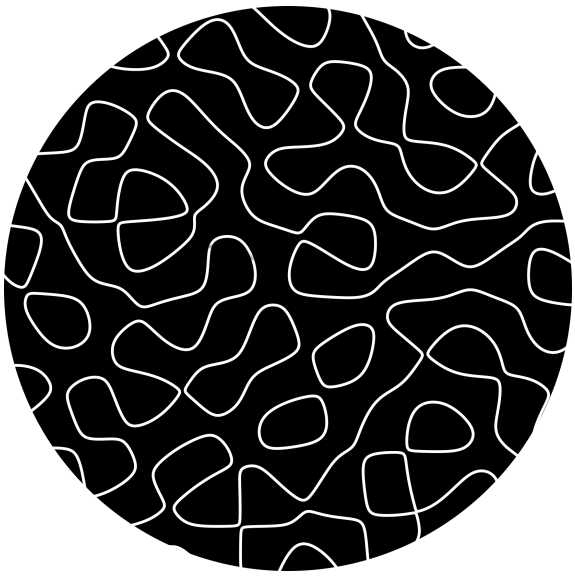

FIGURE 3. Nodal set (white) intersecting a small central disc from the same sample of BARW f_{1885} displayed in Fig. 2.

A useful inequality, conjectured by Rayleigh and later proved independently by Faber and Krahn (cf. [**8**, Chapter IV.2]), is:

LEMMA 2.2 (Faber–Krahn inequality). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Suppose that the Dirichlet eigenvalues are $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \dots$. Then we have*

$$\lambda_1(\Omega) \geq \frac{\pi j_{0,1}^2}{A}$$

where $j_{0,1}$ is the first zero of the Bessel function J_0 and A is the area of Ω. Moreover, equality is obtained if and only if Ω is a disc.

For the BARW f_n the Faber–Krahn inequality implies that each nodal domain Γ satisfies

$$\mathrm{vol}(\Gamma) \geq \frac{\pi j_{0,1}^2}{\lambda_n} \gg \frac{1}{n}, \tag{2.1}$$

which follows from part (vi) of Lemma 2.1.

A trick often employed in the study of Laplace eigenfunctions, is to rewrite the eigenfunction as harmonic function in a space one dimension higher. Suppose $f_n(x)$ is a Laplace eigenfunction for the domain $\mathbb{S}$. Then

$$u_n(x,t) := f_n(x)\cosh(t) \tag{2.2}$$

satisfies

$$\Delta u_n(x,t) = 0$$

on $\mathbb{S} \times \mathbb{R}$. The special properties of harmonic functions such as the mean value theorem and maximum principle can then be brought to bear on eigenfunctions. In particular we have local analytic inequalities of the following form:

LEMMA 2.3 (Local analytic inequalities). *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be smooth and satisfy $\Delta f + \pi^2 n f = 0$, and suppose $\rho > 0$. Then there is a constant $C_\rho > 0$ such that, for any point $x_0 \in \mathbb{R}^2$:*

$$|f(x_0)|^2 \le C_\rho n \int_{B(x_0,\rho/\sqrt{n})} |f(x)|^2\,dx \tag{2.3}$$

$$|\nabla f(x_0)|^2 \le C_\rho n^2 \int_{B(x_0,\rho/\sqrt{n})} |f(x)|^2\,dx \tag{2.4}$$

$$|\nabla\nabla f(x_0)|^2 \le C_\rho n^3 \int_{B(x_0,\rho/\sqrt{n})} |f(x)|^2\,dx \tag{2.5}$$

where $\nabla\nabla f$ denotes the vector of all second partial derivatives, namely:

$$|\nabla\nabla f(x_0)|^2 = (\partial_{x_1}^2 f(x_0))^2 + 2(\partial_{x_1}\partial_{x_2} f(x_0))^2 + (\partial_{x_2}^2 f(x_0))^2.$$

Estimates of this type are classical, but a simple proof using the trick (2.2) is given in [**21**, Appendix B].

REMARK. Since $f : \mathbb{S} \to \mathbb{R}$ satisfying the condition $f|_{\partial\mathbb{S}} = 0$ may be periodically extended to the domain $\mathbb{R}^2$ by reflecting f along the boundary $\partial\mathbb{S}$, these estimates immediately apply to BARWs. In fact, the formulae defining BARWs already encode this extension, because each basis eigenfunction is an odd function in each of its coordinates separately. Thus even when the ball $B(x_0, \rho/\sqrt{n})$ intersects $\partial\mathbb{S}$ these estimates remain valid.

1.2. Global results. We motivated our study of the nodal components of eigenfunctions in Section 1.3 of the introduction by describing the global results

of Courant, Pleijel and Stern, and how wide a range of possible behaviour is left open by them. In Section 2.3 of the same chapter, we introduced a probabilistic model, which turned the nodal component count $N(f)$ into a random variable whose moments we could study.

A further global result concerns the total length of the nodal set of Laplace eigenfunctions and was proved by Donnelly and Fefferman [**13**] in 1990. They provide upper bounds for the length of $Z(f_\lambda)$ in terms of the eigenvalue λ for f_λ a Laplace eigenfunction on a two-dimensional real-analytic Riemannian manifold with boundary.[1] Their work followed that of Brüning and Gromes [**5, 6**], who established the lower bound. We collectively state their results for Laplace eigenfunctions on $\mathbb{S}$.

LEMMA 2.4 (Donnelly–Fefferman, Brüning–Gromes). *Let ϕ_λ be a Dirichlet Laplace eigenfunction on $\mathbb{S}$ with eigenvalue λ, and write $\mathcal{L}(\cdot)$ for the 1-dimensional Hausdorff measure. Then there exist constants $c, C > 0$ such that*

$$c\sqrt{\lambda} \leq \mathcal{L}\left(Z(\phi_\lambda)\right) \leq C\sqrt{\lambda}.$$

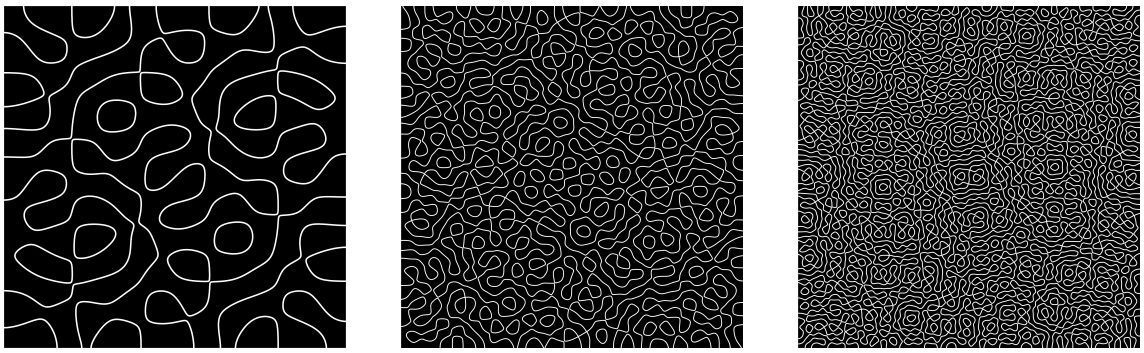

FIGURE 4. Increasingly intricate nodal figures: samples of the BARWs f_{221}, f_{1885} and f_{5525} (left to right).

1.3. Rescaling of BARWs. In light of the local and global properties of Laplace eigenfunctions we have visited so far, together with empirical evidence, we might guess that the 'correct' order of growth of the number of nodal components $N(f_n)$ is roughly on the order of n. Certainly, Courant and Pleijel's theorems imply it can be no greater. It will often be convenient, especially when studying local (stochastic) properties of BARWs, to rescale the variable in the neighbourhood of a

[1] See also [**26**, Theorem 13.15] which gives a more general statement for higher dimensions and under weaker hypotheses.

distinguished point. We define the *scaled* BARW at base-point $x \in \mathbb{S}$ to be

$$f_{x,n}(u) := f_n(x + \tfrac{2u}{\sqrt{n}}) \tag{2.6}$$

where now $u \in B(R)$ for some fixed large parameter $R > 0$ (which should be thought of as $o(n)$). With this parameterisation we should expect that the nodal count in the ball $B(R)$ will be of order a constant (depending upon R).

Remark. A slightly different scaling is usually employed in the study of ARWs, namely $f^{ARW}_{x,n}(u) := f^{ARW}_n(x+\frac{u}{\sqrt{n}})$. The reasoning behind the choices for each model will become clear in Section 1.2, where we see that they enable consistent treatment of their limiting scaled *covariance functions*.

2. Gaussian random fields

Here we will highlight aspects of the theory of Gaussian random fields which pertain to this thesis, making particular reference to BARWs. More comprehensive treatments may be found in the seminal text [**12**]; with emphasis on the geometry in [**1**]; and with key theorems usefully collected in [**18**, Appendix A].

2.1. The covariance function for BARWs. From the perspective of stochastic processes, a BARW is a centred (i.e. zero mean) multivariate Gaussian process, otherwise known as a Gaussian random field. This is a measurable map from a complete (Gaussian) probability space to the space of real-valued functions on a topological space. The covariance function of a Gaussian random field $f : \mathbb{R}^d \to \mathbb{R}$ is given by

$$K_f(x, y) = \mathbb{E}\{f(x)f(y)\}$$

and is an indispensable tool in the study of the statistical properties of f. In fact, the covariance function of a Gaussian random field completely determines the field's distribution; that is, any aspect of the (stochastic) geometry of f is, in principle, expressible in terms of the covariance K_f.

For the particular case of BARWs (1.12), recalling the notation (1.1) and (1.3), the covariance can be expressed as

$$\begin{aligned} K_n(x, y) := K_{f_n}(x, y) &= \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda^{\mathbb{S}}_n} \phi_\lambda(x)\phi_\lambda(y) \\ &= \frac{4}{\mathcal{N}_n} \sum_{\lambda \in \Lambda^{\mathbb{S}}_n} \sin(\pi\lambda_1 x_1) \sin(\pi\lambda_2 x_2) \sin(\pi\lambda_1 y_1) \sin(\pi\lambda_2 y_2) \end{aligned} \tag{2.7}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{S}$. It will also be convenient to define a scaled version of the covariance K_n, as we did for the function f_n with (2.6). We define the *scaled covariance function* at base-point $x \in \mathbb{S}$ to be

$$K_{x,n}(u,v) := K_n(x + \tfrac{2u}{\sqrt{n}}, x + \tfrac{2v}{\sqrt{n}}) = \mathbb{E}\left\{ f_n(x + \tfrac{2u}{\sqrt{n}}) f_n(x + \tfrac{2v}{\sqrt{n}}) \right\} \tag{2.8}$$

where $u, v \in \mathbb{R}^2$. Note that the r.h. equality exhibits $K_{x,n}$ as the (unscaled) covariance of the scaled function $f_{x,n}$. When considering local properties of K_n we will usually restrict u, v to lie within a ball $B(R)$ for some large fixed parameter $R > 0$.

For comparison with (2.7), the covariance function for the ensemble of two-dimensional arithmetic random waves (1.11) is

$$K_n^{ARW}(x,y) = \frac{1}{\#\Lambda_n} \sum_{\lambda \in \Lambda_n} \cos(2\pi\lambda \cdot (x-y)) \tag{2.9}$$

where $x, y \in \mathbb{T}^2$. The covariance function for the random spherical harmonics (1.10) is

$$K_n^{RSH}(x,y) = P_n(\cos(\Theta(x,y))) \tag{2.10}$$

where P_n is the Legendre polynomial of degree n, and $\Theta(x,y)$ is the angle between the vectors x and y on the two-dimensional unit sphere.

2.2. Useful hypotheses on the covariance.

2.2.1. *Translation-invariance and Bochner's theorem.* We say that a smooth Gaussian random function $F : \mathbb{R}^d \to \mathbb{R}$ has *translation-invariant distribution* if for any $m \in \mathbb{N}$, any $u_1, \dots, u_m \in \mathbb{R}^d$, any $v \in \mathbb{R}^d$, the Gaussian random vectors $(F(u_1), \dots, F(u_m))$ and $(F(u_1 + v), \dots, F(u_m + v))$ have the same multivariate normal distribution. For such an F, the covariance kernel $K_F(u,v)$ depends only on the difference $u - v$ and is continuous; that is, there exists a continuous function $k : \mathbb{R}^d \to \mathbb{R}$ such that

$$K_F(u,v) := \mathbb{E}\{F(u)F(v)\} = k(u-v).$$

This covariance is a *positive-definite function*, that is, for any $w_1, \dots, w_m \in \mathbb{R}^d$ the matrix with entries $a_{i,j} := k(w_i - w_j)$ is positive-definite; and so we can make use of a theorem from real analysis:

THEOREM 2.5 (Bochner's Theorem). *A continuous function g on $\mathbb{R}^d$ is positive-definite if and only if there exists a probability measure ρ on $\mathbb{R}^d$ whose Fourier transform is g.*

Thus we see that k may be represented as a Fourier integral

$$k(x) = \int_{\mathbb{R}^d} e^{2\pi i \langle x,\lambda\rangle} d\rho(\lambda), \tag{2.11}$$

where ρ is a positive finite measure which is symmetric with respect to the origin, and called the *spectral measure* of F.

In this translation-invariant setting, provided the variance does not vanish, one may usually assume that F has been normalised so that

$$\mathbb{E}\,|F(u)|^2 = k(0) = \int_{\mathbb{R}^d} d\rho(\lambda) = 1,$$

so that ρ is a probability measure.

The hypothesis of translation-invariance is exceptionally useful and plays a key role in Nazarov and Sodin's Euclidean theorem, upon which their further work – and ours – is based. However it is sometimes too strong a requirement; a weaker hypothesis that is especially useful for ensembles of Gaussian functions is that of translation-invariant local limits.

2.2.2. *Translation-invariant local limits.* Let (f_L) be a family of Gaussian functions defined on the same space X (often described as a *Gaussian ensemble*). The Gaussian ensemble (f_L) is said to have *translation-invariant local limits as* $L \to \infty$ if, for a.e. $x \in X$ there exists a positive-definite even function $k_x : \mathbb{R}^d \to \mathbb{R}$ such that for each $R < \infty$,

$$\lim_{L\to\infty} \sup_{|u||v|\le R} |K_{x,L}(u,v) - k_x(u-v)| = 0. \tag{2.12}$$

If we suppose that the limiting kernels $k_x(u-v)$ do exist, then they are the covariance functions of translation-invariant Gaussian functions $F_x : \mathbb{R}^d \to \mathbb{R}$, each called the *local limiting function* at x. Again, by Bochner's Theorem 2.5, each limiting kernel k_x (normalised so that $k(0) = 1$) is the Fourier transform $\hat{\rho}_x$ of a probability measure ρ_x on $\mathbb{R}^d$ called the *local limiting spectral measure* of the ensemble (f_L) at the point $x \in X$.

To guarantee that the limiting functions F_x make sense and have well-behaved nodal sets, conditions on the 'smoothness' of the covariance functions and conditions on the 'non-degeneracy' of associated covariance matrices are needed.

2.2.3. *Non-degeneracy and covariance matrices.* Understanding low-lying critical points of f_L is important for counting nodal components, for at a critical *zero* x, different nodal lines meet (cf. Lemma 2.1.(iii)–(iv)). Even a very small perturbation

of the eigenfunctions may separate this nodal meeting, causing the nodal count to suddenly increase. Therefore, if there are too many low-lying critical points, the nodal count might be too ill-behaved to estimate.

Normalising a smooth Gaussian function endows it with constant variance, and provided its covariance is continuously differentiable in one of its arguments, we deduce that the jointly Gaussian random variables $f_L(x)$ and $\partial_{x_j} f_L(x)$ are uncorrelated, and thus independent for each j. Thus $f_L(x)$ and $\nabla f_L(x)$ may be treated as independent random vectors, and to understand critical points it suffices to understand the scaled covariance matrix of the gradient vector ∇f_L, which we write as $C_{x,L}(u)$, with entries

$$C_{x,L}(u)_{i,j} = \partial_{u_i}\partial_{v_j} K_{x,L}(u,v)|_{v=u}, \qquad 1 \le i,j \le d.$$

The non-degeneracy condition which suffices as a hypothesis for the important Theorem 2.11 of Nazarov and Sodin described in Section 3, is that: for every $R < \infty$,

$$\varliminf_{L\to\infty} \inf_{x\in X} \inf_{|u|\le R} \{|\det C_{x,L}(u)|\} > 0.$$

For the ensemble of BARWs this condition is not enough: we need to understand the joint distribution of $f_n(x)$ and $\nabla f_n(x)$, since they are no longer independent. We will instead need non-degeneracy of a different scaled covariance matrix $\Sigma_{x,n}(u)$, with entries

$$\Sigma_{x,n}(u)_{i,j} = \partial_{u_i}\partial_{v_j} K_{x,n}(u,v)|_{v=u}, \qquad 0 \le i,j \le 2, \tag{2.13}$$

where an index $i = 0$ or $j = 0$ is to be understood as the zero-th partial derivative. We will see why this matrix is useful later in this chapter through Lemma 2.15, and will prove a non-degeneracy condition for the ensemble of BARWs in Chapter 3.

2.2.4. *Smoothness.* For centred Gaussian functions, local bounds on the smoothness of the covariance function translate into local bounds on the probability that sample functions have large norm. Smoothness in this context is described by bounds on the partial derivatives of the covariance. For Gaussian ensembles we should need bounds which are uniform once the scaling factor is accounted for; namely a Gaussian ensemble (f_L) is said to be C^m-smooth if, for every $R < \infty$,

$$\varlimsup_{L\to\infty} \sup_{x\in X} \sup_{|u|,|v|\le R} \left\{ \left|\partial_u^i \partial_v^j K_{x,L}(u,v)\right| : 0 \le i,j \le m \right\} < \infty.$$

For the specific case of BARWs, this smoothness proves useful because the classical Kolmogorov and Fernique's theorems employed as in [**18**, (A.1)], give that for

each open $U \supset \overline{B}(R)$

$$\mathbb{P}\left\{\|f_{x,n}\|_{\overline{B}(R),1} > A\right\} \ll_{R,U} e^{-c(R,U)A^2/N_{U,2}(K_{x,n})}. \tag{2.14}$$

where $c(R,U)$ is a constant, and where we have used the notation:

$$\|f\|_{Q,r} := \max_{Q} \max_{|\alpha|\leq r} |\partial^\alpha f| = \|f\|_{C^r(Q)}$$
$$N_{V,r}(K) := \max_{|\alpha|,|\beta|\leq r} \sup_{x,y\in V} \left|\partial_x^\alpha \partial_y^\beta K(x,y)\right|.$$

in which Q is compact, and V is an open set containing Q. The upshot is that a uniform (w.r.t. x and n) upper bound on $N_{U,2}(K_{x,n})$ would enable us to choose $A > 0$ so as to make the r.h.s. of (2.14) as small as we desire. To demonstrate that such a bound holds, we explicitly write out the scaled covariance (2.8)

$$K_{x,n}(u,v) = \frac{4}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n^{\mathbb{S}}} \prod_{j=1}^{2} \sin\left(\pi\lambda_j(x_j + \tfrac{2u_j}{\sqrt{n}})\right) \sin\left(\pi\lambda_j(x_j + \tfrac{2v_j}{\sqrt{n}})\right),$$

for which we need a bound on

$$N_{U,2}(K_{x,n}) = \max_{|\alpha|,|\beta|\leq 2} \sup_{u,v\in U} \left|\partial_u^\alpha \partial_v^\beta K_{x,n}(u,v)\right|.$$

where $\partial_u^\alpha = \partial_{u_1}^{\alpha_1}\partial_{u_2}^{\alpha_2}$ and $\alpha, \beta \in \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2)\}$. Using the upper bound of 1 for the size of each sine factor, we see that the first partial derivatives satisfy

$$|\partial_{u_i} K_{x,n}(u,v)| \leq \frac{2\pi\lambda_i}{\sqrt{n}} |K_{x,n}(u,v)| \leq \frac{2\pi\lambda_i}{\sqrt{n}} \frac{4}{\mathcal{N}_n} \mathcal{N}_n \leq 8\pi$$

and, as each second partial derivative yields an addition factor of 2π at most, we conclude that

$$N_{U,2}(K_{x,n}) \leq 4(4\pi^2)^2 = 64\pi^4,$$

holding true for every $x \in \mathbb{S}$, so that the inequality (2.14) gives

$$\mathbb{P}\left\{\|f_{x,n}\|_{\overline{B}(R),1} > A\right\} < C(R)e^{-c(R)A^2}. \tag{2.15}$$

2.3. General Gaussian concentration inequalities. Expressions bearing resemblance to (2.15) but of a different flavour are the concentration inequalities of Bernstein and that of Gaussian measure. These lemmas will play crucial roles in the proof of Theorem 1.2.

The first inequality is the statement that standard Gaussian random vectors of norm greater than two become exponentially unlikely as the dimension of the

Gaussian space grows. This is a classical result which uses little more than the Chebyshev inequality.

LEMMA 2.6 (Bernstein's concentration of norm). *Suppose $X \sim \mathcal{N}(\mathbf{0}, I_n)$. Then there is a positive constant c for which we have*

$$\mathbb{P}\left\{\|X\| > 2\sqrt{n}\right\} \leq e^{-cn}. \tag{2.16}$$

Since the norm of the BARW f_n is equal to the Euclidean norm of the coefficient vector (w.r.t. the orthonormal basis) normalised by $\sqrt{\mathcal{N}_n}$, an immediate corollary for BARWs is

COROLLARY 2.7 (Bernstein for BARWs).

$$\mathbb{P}\left\{\|f_n\| > 2\right\} \leq e^{-c\mathcal{N}_n}.$$

The next result is deeper, coming from an isoperimetric inequality in Gaussian space, proved contemporaneously in the 1970s by Sudakov and Tsirelson [**25**], and by Borell [**4**]. We present the statement specialised to BARWs, once more describing sample functions by their coefficient vectors:

LEMMA 2.8 (Concentration of Gaussian measure). *Suppose that $F \subset \mathcal{H}_n \cong \mathbb{R}^{\mathcal{N}_n}$ is a measurable set, and for a fixed $\rho > 0$ we have $\mathbb{P}\{F_{+\rho}\} < \frac{3}{4}$. Then*

$$\mathbb{P}\{F\} \leq 2e^{-c\rho^2\mathcal{N}_n}.$$

3. Methods for counting nodal components

3.1. Measurability of $N(f)$. To ensure that our study of $N(f_n)$ as a random variable is sensible, it is necessary to verify that $N(f_n)$ is a measurable mapping from the space $\mathcal{H}_n$ equipped with the Gaussian measure, to the natural numbers. A neat result of Rozenshein proves this in a general setting:

LEMMA 2.9 (Measurability of the nodal count [**21**]). *Let X be a compact metric space, let Ω be a (not necessarily complete) probability space, and let $f : X \times \Omega \to \mathbb{R}$ be a random real-valued function on X that is a.s. continuous. Then the number of nodal components of f is a random variable, i.e. a measurable mapping $\Omega \to \mathbb{N} \cup \{0, \infty\}$.*

With $X = \mathbb{S}$, this lemma applies immediately to our situation and therefore $N(f_n)$ is a random variable.

3.2. Asymptotic laws for the number of nodal components. Nazarov–Sodin's paper [**18**] made two key contributions to the counting of nodal components of Gaussian random fields. The first establishes asymptotics for the distribution of the zeroes of translation-invariant smooth Gaussian functions in Euclidean space contained within domains of large volume. The second contribution used techniques developed in the first to prove similar results for ensembles of smooth Gaussian functions on Riemannian manifolds.

3.2.1. *Preliminaries on measures.* Before stating the first result, a few further preliminary definitions for measures are needed. A finite complex-valued measure μ on $\mathbb{R}^d$ is said to be *Hermitian* if $\mu(-E) = \overline{\mu(E)}$ for each Borel set $E \subset \mathbb{R}^d$; the *Fourier transform* $\hat{\mu}$ of the measure μ is

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{2\pi i \langle \xi, \lambda \rangle} d\mu(\lambda).$$

Comparing this latter definition with (2.11), we see that if ρ is the spectral measure of a continuous translation-invariant Gaussian function F, then the covariance function $k = \hat{\rho}$. The *support* of the measure μ is denoted by $\mathrm{spt}(\mu)$, and μ is called *non-atomic* if, for any measurable subset A with $\mu(A) > 0$, there exists a measurable subset B of A such that $\mu(A) > \mu(B) > 0$. Otherwise, μ is called *atomic*.

3.2.2. *Nodal count for translation-invariant Gaussian functions in* $\mathbb{R}^d$. Now denote by $N(R; F)$ the number of connected components of the zero set $Z(F) = F^{-1}\{0\}$ that are contained in the open ball $B(R) = \{x : |x| < R\}$. The count $N(R; F)$ is a random variable and our goal is to describe its behaviour in the limit $R \to \infty$.

THEOREM 2.10 (Nazarov–Sodin – translation-invariant Euclidean case [**18**]). *Suppose that the spectral measure ρ of a continuous Gaussian translation-invariant function F satisfies the following conditions:*

(ρ1)
$$\int_{\mathbb{R}^d} |\lambda|^4 \, d\rho(\lambda) < \infty;$$

(ρ2) *ρ has no atoms;*

(ρ3) *ρ is not supported on a linear hyperplane.*

Then there exists a constant $a_\rho \geq 0$ such that

$$\lim_{R\to\infty} \frac{N(R;F)}{\operatorname{vol} B(R)} = a_\rho \quad a.s. \quad \text{and} \quad \lim_{R\to\infty} \mathbb{E}\left|\frac{N(R;F)}{\operatorname{vol} B(R)} - a_\rho\right| = 0.$$

Furthermore, $a_\rho > 0$ provided that

($\rho 4$) *there exists a finite compactly supported Hermitian measure μ with $spt(\mu) \subset spt(\rho)$ and a bounded domain $D \subset \mathbb{R}^d$ such that $\hat{\mu}|_{\partial D} < 0$ and $(\hat{\mu})_{u_0} > 0$ for some $u_0 \in D$.*

The proof of Theorem 2.10 combines counting techniques from integral-geometry, the stability of components of the zero set under small perturbations, Kac–Rice estimates and fundamental results from ergodic theory. Many of the ideas from their work are present throughout this thesis.

3.2.3. *Nodal count for the Riemannian case.* The results stated here are the motivation for our proof of Theorem 1.1 and its contributory, Lemma 3.4.

THEOREM 2.11 (Nazarov–Sodin – Riemannian case [**18**]). *Suppose that (f_L) is a C^3-smooth non-degenerate Gaussian ensemble on a d-dimensional Riemannian manifold X which has translation-invariant local limits. Suppose that for a.e. $x \in X$ the local limiting spectral measure ρ_x has no atoms. Then there is $\overline{a} \in L^\infty(X)$ for which*

$$\lim_{L\to\infty} \mathbb{E}\left\{\left|L^{-d}N(f_L) - \int_X \overline{a}(x)\, d\mathrm{vol}_X\right|\right\} = 0.$$

A key part in the proof of Theorem 2.11 is the 'local version' below: stating that the local limiting nodal intensities $a_{\rho_x} = \overline{a}(x)$ can be recovered by taking a double limit, first with respect to L, then with respect to R, another (semi-local scaling) parameter.

THEOREM 2.12 (Nazarov–Sodin – local limiting nodal intensity [**18**]). *Under the assumptions of Theorem 2.11, for a.e. $x \in X$ and for every $\epsilon > 0$,*

$$\lim_{R\to\infty} \overline{\lim_{L\to\infty}}\, \mathbb{P}\left\{\left|\frac{N(R;f_{x,L})}{\operatorname{vol} B(R)} - \overline{a}(x)\right| > \epsilon\right\} = 0,$$

where $N(R;f_{x,L})$ is the number of connected components of the zero set $Z(f_L)$ contained in the open ball centred at x in X of radius R/L.

The idea for the Riemannian case, is to attempt to prove that, in the scaling limit $L \to \infty$, we can apply the Euclidean theorem. More precisely, for any fixed $x \in X$ we blow up local coordinates about x to a scale parameter of L, and consider the

scaled Gaussian random function $f_{x,L}$. By the hypotheses, there exists a translation-invariant Gaussian function F_x so that the covariance of $f_{x,L}$ tends to that of F_x as $L \to \infty$. Since the covariance controls all properties of the functions, we might hope to use the asymptotic nodal count properties of F_x to describe the nodal count properties of (f_L) for large L.

Remark. Though the Riemannian case is effective in adapting to a difference in geometry, it can —as highlighted by Nazarov and Sodin— be readily exploited for Gaussian ensembles which are not themselves translation-invariant, but which have some limiting local translation-invariance property. Our result Lemma 3.4 is an adaptation of Theorem 2.12 applied to the ensemble of BARWs, though we do not quite have the limiting property their result requires. We will instead make a mild adaptation of Nazarov and Sodin's proof of Theorem 2.12.

3.3. Integral-geometric sandwich. One of the tools alluded to above as contributing to the proof of Theorem 2.10, is an 'integral-geometric sandwich' inequality. There are a couple of different versions of this, and the version [**17**, Claim 5.1] is most attuned to our situation. We state it here for a non-specific sample[2] f_n of a BARW. Suppose $r > 0$ is some fixed parameter; let $N(x, r; f_n)$ be the number of nodal components of f_n *strictly contained* within the open ball $B(x, r)$; and let $N^*(x, r; f_n)$ be the number of nodal components of f_n which *intersect* the ball $B(x, r)$. In this notation, we have

Lemma 2.13 (Integral-geometric sandwich for BARWs [**17**]).

$$\int_{\mathbb{S}} \frac{N(x, r; f_n)}{\operatorname{vol} B(r)} dx \leq N(f_n) \leq \int_{\mathbb{S}} \frac{N^*(x, r; f_n)}{\operatorname{vol} B(r)} dx. \tag{2.17}$$

The proof of this lemma given in [**17**] is remarkably short and neat, and though stated there for the sphere $\mathcal{S}^2$, the method applies to our situation in identical form. The only complication is that for the square domain $\mathbb{S}$, we have a boundary to contend with. However, the special form of the functions f_n means they may be periodically extended to $[-1, 1]^2$ at which point the domain of f_n is the flat torus, and integration over the boundary presents no difficulties.

The utility of Lemma 2.13 will become evident later through the choice of the parameter r. In the proof of Theorem 1.1 we will use our special knowledge of the properties of Laplace eigenfunctions and introduce another parameter $\delta > 0$, bounding $N^*(x, r; f_n)$ above by $N(x, r + \delta; f_n)$, and giving an effective form of the sandwich with the same type of counting function on each side.

[2]The inequality is deterministic.

3.4. Shell lemma. In this section we describe a fundamental lemma in the existing methods for counting nodal components. Introduced by Nazarov and Sodin [**17**], it is a result from multivariate calculus, and gives quantitative conditions under which a function's nodal count cannot change in a small neighbourhood under small perturbations of the function. We present it here as a combination of Lemmas 4.2 and 4.3 in [**17**]; a proof for higher dimensions is given in [**21**, Appendix B].

LEMMA 2.14 (Shell Lemma [**17**]). *Let $\mathcal{D}$ be a disc, suppose $F \in C^1(\mathcal{D})$, and that at every point $x \in \mathcal{D}$ either $|F(x)| > \mu$ or $|\nabla F(x)| > \nu$. Suppose $G \in C(\mathcal{D})$ and $\sup|G| < \mu$. Then each component $\gamma \in Z(F)$ for which $\operatorname{dist}(\gamma, \partial\mathcal{D}) > \mu/\nu$, generates a component $\tilde{\gamma} \in Z(F+G)$ such that $\tilde{\gamma} \subset \gamma_{+\mu/\nu}$. Moreover, different components $\gamma_1 \neq \gamma_2$ of $Z(F)$ generate different components $\tilde{\gamma_1} \neq \tilde{\gamma_2}$ of $Z(F+G)$.*

The utility of Lemma 2.14 lies in the fact that its hypotheses can be studied probabilistically for Gaussian random waves, enabling us to quantify the probability that the nodal count does not change too drastically. The next section is a local version of this principle.

3.5. Bulinskaya-type lemma. This section is devoted to quantifying the probability that a random eigenfunction's nodal set is 'topologically unstable' in a neighbourhood of a fixed point $x \in \mathbb{S}$. Bulinskaya proved a related qualitative statement [**7**], which was made quantitative in [**18**]. We give a proof of a statement similar to that in the latter, but without the assumption of independence of f and ∇f, making use of a local expansion idea which features repeatedly in their work. Throughout this section we fix a parameter $R > 0$. We will be able to show that, given hypotheses on the determinant of the covariance matrix Σ_x for the joint distribution of $f(x)$ and its gradient $\nabla f(x)$, we can choose $a > 0$ so as to make the probability of 'topological instability' as small as we like. We first consider the event

$$\Omega_a = \left\{ \min_{x \in \overline{B}_R} \max\{|f(x)|, |\nabla f(x)|\} < a \right\},$$

and the lemma we prove is

LEMMA 2.15. *Fix $\delta > 0$ and suppose that*

$$\inf_{x \in B(R+1)} \det(\Sigma_x) \geq \kappa > 0. \tag{2.18}$$

Then we can choose $a > 0$ such that $\mathbb{P}(\Omega_a) < \delta$.

REMARK. Though it is not necessary to write the κ in the statement of this lemma, retaining it will enable immediate deduction of Corollary 2.16 for BARWs, which is our aspiration.

PROOF. We begin deterministically: suppose there is a $u \in B(R)$ for which $|f(u)| < a$ and $|\nabla f(u)| < a$. Then for $\rho \in (0,1)$ a Taylor expansion on the ball $B(u,\rho)$ gives

$$\begin{aligned} |f(z)| \leq A &= |f(u)| + C\,|z-u|\,|\nabla f(u)| \\ &= a + C\rho a, \end{aligned} \tag{2.19}$$

$$|\nabla f(z)| \leq B = C\,|\nabla f(u)| = Ca. \tag{2.20}$$

Therefore, on the event Ω_a we have

$$\mathrm{vol}\left(\{z \in B(R+1) : |f(z)| \leq A, |\nabla f(z)| \leq B\}\right) \geq \mathrm{vol}\left(B(u,\rho)\right),$$

and thus

$$\begin{aligned} \mathbb{P}(\Omega_a) = \mathbb{E}\{\mathbf{1}_{\Omega_a}\} &\leq \mathbb{E}\left\{\frac{\mathrm{vol}\left(\{z \in B(R+1) : |f(z)| \leq A, |\nabla f(z)| \leq B\}\right)}{\pi\rho^2}\right\} \\ &= \tfrac{1}{\pi\rho^2}\mathbb{E}\left\{\int_{B(R+1)} \mathbf{1}_{|f|\leq A, |\nabla f| \leq B}(z)dz\right\} \\ &= \tfrac{1}{\pi\rho^2}\int_{B(R+1)} \mathbb{E}\left\{\mathbf{1}_{|f|\leq A, |\nabla f| \leq B}(z)\right\} dz \\ &\leq \tfrac{\mathrm{vol}\, B(R+1)}{\pi\rho^2} \sup_{z \in B(R+1)} \mathbb{E}\left\{\mathbf{1}_{|f|\leq A, |\nabla f| \leq B}(z)\right\}, \end{aligned} \tag{2.21}$$

which reduces to an explicit computation of probability for the Gaussian random vector $(f(z), \nabla f(z))$, whose distribution we understand through its covariance matrix Σ_z. For each fixed z, we have

$$\begin{aligned} \mathbb{E}\left\{\mathbf{1}_{|f|\leq A, |\nabla f| \leq B}(z)\right\} &= \int_{\mathbb{R}^3} \frac{\exp\left(-\frac{1}{2}y(\Sigma_z)^{-1}y^T\right)}{(2\pi)^{3/2}\sqrt{\det \Sigma_z}} \mathbf{1}_{|y_1|\leq A, |(y_2,y_3)| \leq B}(z)dy \\ &\leq \frac{(2A)(\pi B^2)}{\sqrt{\det(\Sigma_z)}}, \end{aligned} \tag{2.22}$$

but the bound (2.18) is uniform in z, and using this with (2.21) and (2.22) we get

$$\mathbb{P}(\Omega_a) \leq \tfrac{\mathrm{vol}\, B(R+1)}{\kappa^{1/2}\rho^2}(2A)B^2.$$

Using the Taylor estimates (2.19) and (2.20) for A and B resp. gives the bound

$$\mathbb{P}(\Omega_a) \leq \tfrac{\mathrm{vol}\, B(R+1)}{\kappa^{1/2}\rho^2} a(1 + C_1\rho)C_2 a^2$$

and if we chose $\rho = a$, then the numerator is of order a^3 and the denominator is of order a^2. That is:

$$\mathbb{P}(\Omega_a) \leq C_{R,\kappa}\, a,$$

and we can choose $a \in (0,1)$ as small as necessary so as to make the r.h.s. smaller than any fixed $\delta > 0$. □

If we have a further condition on the local smoothness of the covariance function K_f, then the discussion in Section 2.2.4 enables us to make a stronger statement regarding the event

$$\Omega_a^A := \left\{ \|f\|_{C^1(B(R)),1} > A \quad \text{or} \quad \min_{x \in \overline{B}_R} \max\{|f(x)|, |\nabla f(x)|\} < a \right\}.$$

This will play a crucial role at the beginning of our proof of Lemma 3.4. Using (2.15) for the particular case of BARWs, the following is an immediate consequence of Lemma 2.15:

COROLLARY 2.16. *Fix $\delta > 0$, fix $x \in \mathbb{S}$, and suppose that the condition* (2.18) *holds uniformly (w.r.t. n) for $f_{x,n}$: that is, there exists $\kappa > 0$ such that*

$$\inf_{x \in B(R+1)} \det(\Sigma_{x,n}) \geq \kappa \qquad \forall n.$$

Then we can choose $a > 0$ small and A large so that $\mathbb{P}\left(\Omega_a^A\right) < \delta$.

4. Ancillary lemmas

In this section we present two simple lemmas which do not readily fit into any of the preceding sections. The second, Lemma 2.18, will be a useful tool in establishing alternative hypotheses for BARWs, enabling us to go from quantitative estimates for the vanishing of a second moment, to quantitative estimates for the vanishing of a supremum. Though we present our own proofs, the results almost certainly exist in the literature, implicitly if not explicitly.

4.1. Square-to-sup lemma. To establish our desired properties of BARWs we will combine Markov's inequality with the following lemma. We recall the notation $\|g\|_{\mathcal{D}}^{\infty} := \sup_{w\in\mathcal{D}} |g(w)|$ and

$$\|g\|_{\mathcal{D}}^{Lip} = \sup_{w,v\in\mathcal{D}} \frac{|g(w)-g(v)|}{|w-v|}.$$

LEMMA 2.17. *Fix $R > 0$, fix $d \in \mathbb{N}$ and let $\mathcal{R} = [-R, R]^d \subset \mathbb{R}^d$. Let (f_L) be a sequence of (deterministic) real-valued continuously-differentiable functions defined on $\mathcal{R}$, with Lipschitz constant $\|f_L\|_{\mathcal{R}}^{Lip} \leq D$ bounded uniformly with respect to R and L. If*

$$\int_{\mathcal{R}} f_L(u)^2 du \to 0 \qquad (L \to \infty),$$

then

$$\sup_{\mathcal{R}} |f_L| \to 0 \qquad (L \to \infty). \tag{2.23}$$

REMARK. The proof given below for Lemma 2.17 will be re-purposed for the quantitative Lemma 2.18, though we note that a simpler argument would probably suffice for the qualitative statement (2.23). It will be convenient to prove for a general d, since we will apply the result in two different cases in order to deduce Corollary 3.2 from Lemma 3.1.

PROOF. Partition $\mathcal{R}$ into boxes $\{B_j\}$ of equal sidelength δ (to be chosen shortly). Let $N_A(\delta)$ be the number of the B_j which contain a point x at which $|f_L(x)| \geq A$, for A some positive parameter. For all y in each such box B_j we have that

$$|f_L(y)| \;\geq\; |f_L(x)| - D\,\mathrm{diam}(B_j) \;\geq\; A - D\delta\sqrt{d}. \tag{2.24}$$

Since D is uniform in R and L, for each fixed A, we can choose δ to be sufficiently small so that for every point in each of the $N_A(\delta)$ boxes the r.h.s. of (2.24) is positive. Thus, taking the integral of $(f_L(u))^2$ over any of the B_j gives

$$\int_{B_j} f_L(u)^2 du \geq \mathrm{meas}(B_j)\cdot(A-\delta D\sqrt{d})^2 = \delta^d\cdot(A-\delta D\sqrt{d})^2 > 0 \tag{2.25}$$

We make the choice $\delta = \frac{A}{2D\sqrt{d}}$ so that

$$\int_{B_j} f_L(u)^2 du \geq \frac{A^{d+2}}{2^{d+2}d^{d/2}D^d}. \tag{2.26}$$

But now integrating over the whole domain:

$$\int_{\mathcal{R}} f_L(u)^2 du \geq N_A \frac{A^{d+2}}{2^{d+2} d^{d/2} D^d} \tag{2.27}$$

and by hypothesis, the l.h.s. of (2.27) may be made arbitrarily small by choosing sufficiently large L. We choose L such that the l.h.s. of (2.27) is less than $\frac{A^{d+2}}{2^{d+2} d^{d/2} D^d}$, and conclude that the non-negative integer N_A must equal zero. Hence for any fixed $A > 0$, for sufficently large L, we have $\sup_{\mathcal{R}} |f_L| \leq A$, which is the claim (2.23). □

4.2. Mean-square to mean-sup lemma.

LEMMA 2.18. *Now let (f_L) be a sequence of continuously-differentiable functions $f_L : \mathbb{S} \times \mathcal{R} \to \mathbb{R}$ and write $f_{x,L}(u) := f_L(x,u)$. Suppose that $\|f_L\|^{Lip}_{\mathbb{S}\times\mathcal{R}} \leq D$ is bounded uniformly with respect to R and L and that*

$$\int_{\mathbb{S}} \int_{\mathcal{R}} \left(f_{x,L}(u)\right)^2 du dx \ll \frac{R^d}{\lambda(L)}, \tag{2.28}$$

where $\lambda(L)$ is some positive real function of L. Then

$$\int_{\mathbb{S}} \|f_{x,L}\|^\infty_{\mathcal{R}} \, dx \ll \frac{R}{(\lambda(L))^{1/d+3}}. \tag{2.29}$$

PROOF. Fix $\delta > 0$ a parameter to be chosen later; define:

$$\mathbb{S}_\delta = \left\{ x \in \mathbb{S} : \int_{\mathcal{R}} \left(f_{x,L}(u)\right)^2 du < \delta \right\}.$$

By using Markov's inequality with (2.28), for any fixed $\delta > 0$

$$\text{meas}(\mathbb{S} \setminus \mathbb{S}_\delta) \leq \frac{1}{\delta} \int_{\mathbb{S}} \left(\int_{\mathcal{R}} \left(f_{x,L}(u)\right)^2 du \right) dx \ll \frac{1}{\delta} \frac{R^d}{\lambda(L)}.$$

We partition the domain of integration of $\|f_{x,L}\|^\infty_{\mathcal{R}}$ into $\mathbb{S}_\delta$ and its complement $\mathbb{S} \setminus \mathbb{S}_\delta$. For the integral over the set $\mathbb{S} \setminus \mathbb{S}_\delta$ we use the uniform bound $\|f_{x,L}\|^\infty_{\mathcal{R}} \ll 1$ to give

$$\int_{\mathbb{S}} \|f_{x,L}\|^\infty_{\mathcal{R}} dx \leq \int_{\mathbb{S}_\delta} \|f_{x,L}\|^\infty_{\mathcal{R}} dx + O\left(\frac{1}{\delta} \frac{R^d}{\lambda(L)} \right). \tag{2.30}$$

To estimate the remaining integral on the r.h.s. of (2.30) we appeal to (2.27), setting $A = \|f_{x,L}\|^\infty_{\mathcal{R}}$ and with $f_L(u)$ replaced with $f_{x,L}(u)$. Because $\mathcal{R}$ is compact and $f_{x,L}$ is C^1, the supremum $A = \|f_{x,L}\|^\infty_{\mathcal{R}}$ is achieved and thus $N_A \geq 1$, which gives the inequality

$$\int_{\mathcal{R}} (f_{x,L}(u))^2 du \geq \frac{(\|f_{x,L}\|^\infty_{\mathcal{R}})^{d+2}}{2^{d+2} d^{d/2} D^d}. \tag{2.31}$$

For each $x \in \mathbb{S}_\delta$, the l.h.s. of (2.31) is bounded above by δ. Rearranging this and recalling our running assumption that $D \ll 1$, we have

$$\|f_{x,L}\|_{\mathcal{R}}^{\infty} < \left(\delta 2^{d+2} d^{d/2} D^d\right)^{\frac{1}{d+2}} \ll_d \delta^{\frac{1}{d+2}}$$

uniformly for $x \in \mathbb{S}_\delta$, and thus

$$\int_{\mathbb{S}_\delta} \|f_{x,L}\|_{\mathcal{R}}^{\infty} dx \ll_d \delta^{\frac{1}{d+2}} \operatorname{meas} \mathbb{S}_\delta \ll \delta^{\frac{1}{d+2}}.$$

Using this in (2.30) we get

$$\int_{\mathbb{S}} \|f_{x,L}\|_{\mathcal{R}}^{\infty} dx \ll_d \delta^{\frac{1}{d+2}} + \frac{1}{\delta} \frac{R^d}{\lambda(L)},$$

and choosing $\delta^{\frac{1}{d+2}} = (R/\lambda(L))^{\frac{1}{d+3}}$ we conclude (2.29). □

CHAPTER 3

Developments

There are marked differences between BARWs and the arithmetic random waves, random spherical harmonics, and Gaussian random functions which the paper [**18**] was designed to tackle. Foremost are both the non-stationarity of the ensemble of BARWs, and the possible degeneracy of the distribution of the random vectors $(f_n(x), \nabla f_n(x))$ which arises from the determinism-inducing Dirichlet boundary condition. In the first section of this chapter we will further illustrate and discuss these differences; in the second section we show that slightly weaker stationarity and non-degeneracy conditions do hold for BARWs; and in the final section we adapt a result from [**18**] to show that there exists a local limiting nodal intensity which may be extracted by means of a double scaling limit.

1. Complexities arising in the study of BARWs

1.1. Non-stationarity. The scaled covariance function $K_{x,n}$ for BARWs is

$$K_{x,n}(u,v) = \frac{4}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n^{\mathbb{S}}} \prod_{j=1}^{2} \sin\left(\pi\lambda_j(x_j + \tfrac{2u_j}{\sqrt{n}})\right) \sin\left(\pi\lambda_j(x_j + \tfrac{2v_j}{\sqrt{n}})\right), \tag{3.1}$$

and here mild complications already exhibit themselves. In each of the previous studies described in Chapter 1.2, it is immediately obvious that the scaled covariance function is translation-invariant as a function of (u, v), depending only on the difference $u - v$. For comparison, recall (2.9) and (2.10). For BARWs however, there is blatant dependency of the scaled covariance on the base-point $x = (x_1, x_2)$, and moreover, there are base-points for which the covariance is demonstrably not translation-invariant. This is easiest to exhibit on the boundary: take $z \in \partial\mathbb{S}$ and consider the variance by setting $v = u$. If K_n were translation invariant we should —at the very least— expect that varying u would not change the value of $K_{z,n}(u,u)$. However

$$K_{z,n}(u,u) = \frac{4}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n^{\mathbb{S}}} \sin\left(2\pi\lambda_1(\tfrac{u_1}{\sqrt{n}})\right)^2 \sin\left(2\pi\lambda_2(\tfrac{u_2}{\sqrt{n}})\right)^2, \tag{3.2}$$

and in particular, $K_{z,n}(0,0) = 0$. Translation-invariance would then imply that $K_{z,n}(u,u)$ is zero everywhere; but this is not the case: we can always observe a positive value of $K_{z,n}(u,u)$ by picking each u_j so as not to be an integer multiple of $\frac{\sqrt{n}}{2\lambda_j}$ for any $\lambda \in \Lambda_n^{\mathbb{S}}$. This guarantees that at least one of the summands on the r.h.s. of (3.2) has both sines non-vanishing, so that $K_{z,n}(u,u) > 0$. If the variance were always positive, then it might be possible to resolve this issue by re-normalisation; however since $K_{z,n}(u,u) = 0$ may also vanish at points *within* the domain $\mathbb{S}$, this avenue is not available to us.

Instead, to grasp how close $K_{x,n}$ is to translation-invariance, we rewrite (3.1) using trigonometric double-angle identities

$$K_{x,n}(u,v) := K_{x,n}^{\dagger}(u,v) - \epsilon_{x,n}(u,v),$$

where we have a translation-invariant (and base-point independent[1]) part:

$$K_{x,n}^{\dagger}(u-v) := K_{x,n}^{\dagger}(u,v) = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n^{\mathbb{S}}} \cos\left(2\pi\lambda_1(\tfrac{u_1-v_1}{\sqrt{n}})\right) \cos\left(2\pi\lambda_2(\tfrac{u_2-v_2}{\sqrt{n}})\right); \tag{3.3}$$

and a translation-dependent part:

$$\begin{aligned} \epsilon_{x,n}(u,v) := \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n^{\mathbb{S}}} (&\cos\left(2\pi\lambda_1(x_1 + \tfrac{u_1+v_1}{\sqrt{n}})\right) \cos\left(2\pi\lambda_2(\tfrac{u_2-v_2}{\sqrt{n}})\right) \\ &+ \cos\left(2\pi\lambda_1(\tfrac{u_1-v_1}{\sqrt{n}})\right) \cos\left(2\pi\lambda_2(x_2 + \tfrac{u_2+v_2}{\sqrt{n}})\right) \\ &- \cos\left(2\pi\lambda_1(x_1 + \tfrac{u_1+v_1}{\sqrt{n}})\right) \cos\left(2\pi\lambda_2(x_2 + \tfrac{u_2+v_2}{\sqrt{n}})\right)). \end{aligned} \tag{3.4}$$

1.2. The translation-invariant part. Making use of the symmetries of Λ_n we can rewrite (3.3) as

$$K_{x,n}^{\dagger}(u-v) = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \cos\left(2\pi\lambda \cdot (\tfrac{u-v}{\sqrt{n}})\right),$$

which exhibits it as the scaled covariance function for (two-dimensional) arithmetic random waves (2.9), namely

$$K_{x,n}^{\dagger}(w) = K_{x,n}^{ARW}(w).$$

However, as with arithmetic random waves, we only obtain convergence to a limiting covariance kernel by restricting to a parametric sub-family of the ensemble, which explains the hypothesis in the statement of Theorem 1.1. We now examine this.

[1]We retain a notational subscript of x so as to indicate that we are considering the *scaled* covariance, despite the part $K_{x,n}^{\dagger}$ happening to be independent of x.

Fix a subsequence (n_j) so that the induced discrete probability measure μ_{n_j} tends weakly (as $j \to \infty$) to the probability measure ν on the unit circle $\mathcal{S}^1$. Then for each fixed $u, v \in B(R)$, writing $w = u - v$, we have that as $j \to \infty$,

$$K^{\dagger}_{x,n_j}(w) = \frac{1}{\mathcal{N}_{n_j}} \sum_{\lambda \in \Lambda_{n_j}} \cos\left(2\pi \tfrac{\lambda}{\sqrt{n_j}} \cdot w\right) \to \int_{S^1} \cos(2\pi\xi \cdot w) d\nu(\xi) =: k_\nu(w). \tag{3.5}$$

This 'target' limiting covariance is visibly even: $k_\nu(u - v) = k_\nu(v - u)$; and is also the Fourier (cosine-) transform of the probability measure ν so that Theorem 2.5 implies that k_ν is a positive-definite function.

Thus (3.5) demonstrates that the translation-invariant part has a local limit in the sense of (2.12) at every point $x \in \mathbb{S}$, and moreover the local limiting covariance function is independent of x.

1.3. The translation-dependent part. We turn our attention to (3.4). In lieu of translation-invariance, we might still hope to have translation-invariant local limits (2.12), under the hypothesis of which direct application of methods from the previous chapter's Section 3 is possible. However, it is not obvious how to prove this – even with control on the limiting distribution of the lattice point set Λ_n – since we would need to make strong statements about arithmetic trigonometric sums for specific fixed points x. In Section 2.1 we instead prove the weaker statement that, when averaged over $\mathbb{S}$, most points tend towards local uniform translation-invariance. We then adapt the wider nodal counting methodology to account for this altered assumption in Section 3 and Chapter 4.

1.4. Degeneracy. The vanishing of the covariance brings with it an additional problem: since BARWs are centred Gaussian random fields, a vanishing covariance implies a deterministic nodal point. More than that, it is no longer possible to treat f_n and its gradient vector ∇f_n as being independent random vectors – a treatment which played a recurrent underlying role in Chapter 2.3. We must focus instead on how they are jointly distributed, which is described locally by the covariance matrix $\Sigma_{x,n}$ (cf (2.13)). As we saw in the Bulinskaya-type lemma, the appropriate non-degeneracy condition is a scaled version of (2.18):

$$\inf_{u \in B(R+1)} \det\left(\Sigma_{x,n}(u)\right) \geq \kappa > 0, \tag{3.6}$$

which we would need to hold uniformly for a.e. $x \in \mathbb{S}$. It is not obvious that such a condition can hold: four deterministic critical zeroes exhibit themselves at the corners of $\mathbb{S}$ for every n, and there may be many more critical zeroes or low-lying critical points which are difficult to specify. Again, the dependence of the BARW

covariance on the base-point makes the condition (3.6) challenging to calculate for specific points x. Instead we resolve this issue indirectly in Section 2.

2. Alternative hypotheses for BARWs

2.1. Locally uniform limiting lemmas. In this section we will make repeated use of Lemma 2.18 in order to demonstrate that various scaled covariance properties of BARWs approach some 'limiting property' locally uniformly (over a ball $B(R)$) on a large proportion of the base-points $x \in \mathbb{S}$. The two flavours of covariance property which we will consider are those mentioned in Section 1: the first quantifies how close the covariance $K_{x,n}(u,v)$ is to local uniform translation-invariance; the second quantifies how close the joint distribution of the random vectors $(f_n(x), \nabla f_n(x))$ are to local uniform non-degeneracy.

LEMMA 3.1. *For $R > 0$ set $\mathcal{R}^4 = [-R,R]^2 \times [-R,R]^2$. Then*

$$\int_{\mathbb{S}} \int_{\mathcal{R}^4} \left(K_{x,n}(u,v) - K^{\dagger}_{x,n}(u-v)\right)^2 du dv\, dx \ll \frac{R^4}{\mathcal{N}_n}. \tag{3.7}$$

For $R > 0$ set $\mathcal{R}^2 = [-R,R]^2$. Then

$$\int_{\mathbb{S}} \int_{\mathcal{R}^2} \left(4\pi^4 - \det \Sigma_{x,n}(u)\right)^2 du\, dx \ll \frac{R^2}{\mathcal{N}_n}. \tag{3.8}$$

By taking each of these in turn as the hypothesis (2.28) of Lemma 2.18, we immediately deduce:

COROLLARY 3.2.

$$\int_{\mathbb{S}} \left\| K_{x,n} - K^{\dagger}_{x,n} \right\|^{\infty}_{\mathcal{R}^4} dx \ll \frac{R}{\mathcal{N}_n^{\frac{1}{7}}}, \tag{3.9}$$

$$\int_{\mathbb{S}} \left\| 4\pi^4 - \det \Sigma_{x,n} \right\|^{\infty}_{\mathcal{R}^2} dx \ll \frac{R}{\mathcal{N}_n^{\frac{1}{5}}}. \tag{3.10}$$

REMARK. The proof of both parts of Lemma 3.1 feature simple but involved calculations of trigonometric integrals, and in both cases it will be useful to first compute the integral w.r.t. x for a fixed u (and also fixed v in the case of (3.7)), and then integrate this expression w.r.t. u, v to obtain the final results. In fact, regarding (3.8) we have a uniform estimate for *each* fixed $u \in \mathbb{R}^2$ of

$$\int_{\mathbb{S}} \left(4\pi^4 - \det \Sigma_{x,n}(u)\right)^2 dx \ll \frac{1}{\mathcal{N}_n} \tag{3.11}$$

so that (3.8) is simply an integration of a constant expression over the square $[-R,R]^2$ which has volume of order R^2. We will make additional use of this pointwise estimate in the proof of Theorem 1.2, there taking $u=0$.

REMARK. We also note that the non-degeneracy constant of $4\pi^4$ in (3.8) and (3.10) coincides (as it should) with the determinant of the covariance matrix $C_{x,n}(u)$ for two-dimensional ARWs as computed in [**21**, Section 3].

PROOF OF LEMMA 3.1 (3.7). *STEP 1: Fixed u,v, averaged w.r.t. x:* With $u,v\in[-R,R]^2$ fixed, recalling the notation (3.4),

$$\int_{\mathbb{S}}(\epsilon_{x,n}(u,v))^2dx=\frac{1}{\mathcal{N}_n^2}\sum_{\lambda\in\Lambda_n^{\mathbb{S}}}\sum_{\mu\in\Lambda_n^{\mathbb{S}}}\sum_{1\le i,j\le 3}\int_{\mathbb{S}}C_i(\lambda,x)C_j(\mu,x)dx \tag{3.12}$$

where, so as to overcome limitations imposed by the margin, we write

$$\begin{aligned}
C_1(\lambda,x)&:=\cos\left(2\pi\lambda_1(x_1+\tfrac{u_1+v_1}{\sqrt{n}})\right)\cos\left(2\pi\lambda_2(\tfrac{u_2-v_2}{\sqrt{n}})\right)\\
C_2(\lambda,x)&:=\cos\left(2\pi\lambda_1(\tfrac{u_1-v_1}{\sqrt{n}})\right)\cos\left(2\pi\lambda_2(x_2+\tfrac{u_2+v_2}{\sqrt{n}})\right)\\
C_3(\lambda,x)&:=\cos\left(2\pi\lambda_1(x_1+\tfrac{u_1+v_1}{\sqrt{n}})\right)\cos\left(2\pi\lambda_2(x_2+\tfrac{u_2+v_2}{\sqrt{n}})\right).
\end{aligned}$$

The computation of the r.h.s. of (3.12) simplifies substantially, because: we can treat each integral over the integrands C_iC_j separately and each factorises w.r.t. its coordinates; the factors without dependence on x are not integrated; and the coordinates of all lattice points in $\Lambda_n^{\mathbb{S}}$ have non-zero λ_i, so that we only end up considering integrals of cosines or squares of cosines over their respective periods. Thus the only non-zero integrals on the r.h.s. of (3.12) are:

$$\int_{\mathbb{S}}(C_1(\lambda,x))^2dx=\tfrac{1}{2}\cos^2\left(2\pi\lambda_2\frac{u_2-v_2}{\sqrt{n}}\right),$$

$$\int_{\mathbb{S}}(C_2(\lambda,x))^2dx=\tfrac{1}{2}\cos^2\left(2\pi\lambda_1\frac{u_1-v_1}{\sqrt{n}}\right),$$

and

$$\int_{\mathbb{S}}(C_3(\lambda,x))^2dx=\tfrac{1}{4};$$

and so we conclude that for each fixed pair $u,v\in[-R,R]^2$:

$$\int_{\mathbb{S}}(\epsilon_{x,n}(u,v))^2dx=\frac{1}{\mathcal{N}_n^2}\sum_{\lambda\in\Lambda_n^{\mathbb{S}}}\left(\tfrac{1}{2}\cos^2\left(2\pi\lambda_1\frac{u_1-v_1}{\sqrt{n}}\right)+\tfrac{1}{2}\cos^2\left(2\pi\lambda_2\frac{u_2-v_2}{\sqrt{n}}\right)+\tfrac{1}{4}\right).$$

STEP 2: Averaged w.r.t. u, v and x. Now we turn to proving (3.7), and begin by exchanging the order of integration on its l.h.s. then inputting the expression in the line above, so that

$$\int_{\mathbb{S}} \left(\int_{\mathcal{R}^4} \left(\epsilon_{x,n}(u,v)\right)^2 du dv \right) dx$$

(3.13)

$$= \int_{\mathcal{R}^4} \left(\frac{1}{\mathcal{N}_n^2} \sum_{\lambda \in \Lambda_n^{\mathbb{S}}} \left(\tfrac{1}{2} \cos^2 \left(2\pi\lambda_1 \tfrac{u_1 - v_1}{\sqrt{n}} \right) + \tfrac{1}{2} \cos^2 \left(2\pi\lambda_2 \tfrac{u_2 - v_2}{\sqrt{n}} \right) + \tfrac{1}{4} \right) \right) du dv$$

We integrate each summand separately, first those whose argument contains λ_1:

$$\begin{aligned} 2 \int_{\mathcal{R}^2} \cos^2 \left(2\pi\lambda_1 (\tfrac{u_1 - v_1}{\sqrt{n}}) \right) du_1 dv_1 &= \int_{\mathcal{R}^2} \left(\cos \left(4\pi\lambda_1 (\tfrac{u_1 - v_1}{\sqrt{n}}) \right) + 1 \right) du_1 dv_1 \\ &= \int_{\mathcal{R}} \frac{\sqrt{n}}{4\pi\lambda_1} \left(\sin \left(4\pi\lambda_1 \tfrac{R - v_1}{\sqrt{n}} \right) - \sin \left(4\pi\lambda_1 \tfrac{-R - v_1}{\sqrt{n}} \right) \right) dv_1 + 4R^2 \\ &= \left(\frac{\sqrt{n}}{4\pi\lambda_1} \right)^2 \left((1 - \cos(4\pi\lambda_1 \tfrac{2R}{\sqrt{n}})) - (\cos(4\pi\lambda_1 \tfrac{2R}{\sqrt{n}}) - 1) \right) + 4R^2 \\ &= \left(\frac{\sqrt{n}}{4\pi\lambda_1} \right)^2 \left(2 - 2\cos(8\pi\lambda_1 \tfrac{R}{\sqrt{n}}) \right) + 4R^2. \end{aligned}$$

Integrating this latter line w.r.t. u_2 and v_2, we simply accumulate a factor of $4R^2$:

$$\begin{aligned} \int_{\mathcal{R}^2} \int_{\mathcal{R}^2} \tfrac{1}{2} \cos^2 \left(2\pi\lambda_1 (\tfrac{u_1 - v_1}{\sqrt{n}}) \right) du dv &= 4R^2 \tfrac{1}{4} \left(\left(\frac{\sqrt{n}}{4\pi\lambda_1} \right)^2 \left(2 - 2\cos(8\pi\lambda_1 \tfrac{R}{\sqrt{n}}) \right) + 4R^2 \right) \\ &= \tfrac{1}{8} \left(\frac{R\sqrt{n}}{\pi\lambda_1} \right)^2 \left(1 - \cos(8\pi\lambda_1 \tfrac{R}{\sqrt{n}}) \right) + 4R^4. \end{aligned}$$

Precisely the same calculation works for those whose argument contains λ_2, and so we have deduced that (3.13) equals:

$$\frac{1}{\mathcal{N}_n^2} \sum_{\lambda \in \Lambda_n^{\mathbb{S}}} \left(\tfrac{1}{8} \left(\frac{R\sqrt{n}}{\pi\lambda_1} \right)^2 (1 - \cos(8\pi\lambda_1 \tfrac{R}{\sqrt{n}})) + \tfrac{1}{8} \left(\frac{R\sqrt{n}}{\pi\lambda_2} \right)^2 (1 - \cos(8\pi\lambda_2 \tfrac{R}{\sqrt{n}})) + 12R^4 \right)$$

which, in order to analyse, we consider separately as the three independent sums:

$$\frac{1}{\mathcal{N}_n^2} \sum_{\lambda \in \Lambda_n^{\mathbb{S}}} \tfrac{1}{8} \left(\frac{R\sqrt{n}}{\pi\lambda_1} \right)^2 (1 - \cos(8\pi\lambda_1 \tfrac{R}{\sqrt{n}})), \tag{3.14}$$

$$\frac{1}{\mathcal{N}_n^2} \sum_{\lambda \in \Lambda_n^{\mathbb{S}}} \tfrac{1}{8} \left(\frac{R\sqrt{n}}{\pi\lambda_2} \right)^2 (1 - \cos(8\pi\lambda_2 \tfrac{R}{\sqrt{n}})), \tag{3.15}$$

and

$$\frac{1}{\mathcal{N}_n^2}\sum_{\lambda\in\Lambda_n^{\mathbb{S}}} 12R^4. \tag{3.16}$$

The third sum, (3.16), is easy to calculate:

$$\frac{1}{\mathcal{N}_n^2}\sum_{\lambda\in\Lambda_n^{\mathbb{S}}} 12R^4 = \frac{12R^4}{\mathcal{N}_n}.$$

The sums (3.14) and (3.15) are equal, exhibited by the symmetry $(\lambda_1,\lambda_2)\to(\lambda_2,\lambda_1)$ of the set of $\Lambda_n^{\mathbb{S}}$. For the first, (3.14), pulling out a factor independent of the λ we are left to consider the size of the sum

$$E_R(n) := \sum_{\lambda\in\Lambda_n^{\mathbb{S}}} \frac{1}{\lambda_1^2}\left(1-\cos(8\pi R\tfrac{\lambda_1}{\sqrt{n}})\right). \tag{3.17}$$

Using an explicit error for the Taylor expansion of $\cos x$ at $x=0$, we have

$$1-\cos x \le \frac{x^2}{2} \qquad \forall x,$$

and so

$$E_R(n) \le \sum_{\lambda\in\Lambda_n^{\mathbb{S}}} \frac{1}{\lambda_1^2}\frac{(8\pi R\frac{\lambda_1}{\sqrt{n}})^2}{2} = \mathcal{N}_n\frac{32\pi^2R^2}{n}.$$

Re-attributing the factor extracted prior to the line (3.17), we see that the sum (3.14) has size

$$\frac{R^2n}{8\pi^2\mathcal{N}_n^2}E_R(n) \le 4\frac{R^4}{\mathcal{N}_n}.$$

Thus (3.14), (3.15) and (3.16) are each individually of size $O(R^4/\mathcal{N}_n)$, and we have proved the claim (3.7) that

$$\int_{\mathbb{S}}\int_{\mathcal{R}^4} \left(\epsilon_{x,n}(u,v)\right)^2 du dv\, dx \ll \frac{R^4}{\mathcal{N}_n}.$$

□

The proof of the second part of Lemma 3.1 is still more involved, on account of the additional trigonometric factors contributing to the determinant of $\Sigma_{x,n}$. We choose temporary non-standard notation so as the make the calculations more readable.

PROOF OF LEMMA 3.1 (3.8). *STEP 0: Non-standard notation.*
Begin by defining

$$\begin{aligned} ss(\lambda, x) &:= \sin(\pi\lambda_1 x_1)\sin(\pi\lambda_2 x_2) \\ sc(\lambda, x) &:= \sin(\pi\lambda_1 x_1)\cos(\pi\lambda_2 x_2) \\ cs(\lambda, x) &:= \cos(\pi\lambda_1 x_1)\sin(\pi\lambda_2 x_2) \\ cc(\lambda, x) &:= \cos(\pi\lambda_1 x_1)\cos(\pi\lambda_2 x_2), \end{aligned}$$

and let $y := x + \frac{2u}{\sqrt{n}}$. With this notation, the entries (2.13) of the scaled covariance matrix $\Sigma_{x,n}(u)$ are

$$\begin{aligned} \left(\Sigma_{x,n}(u)\right)_{00} &= \tfrac{4}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n^{\mathbb{S}}} ss(\lambda, y)^2 \\ \left(\Sigma_{x,n}(u)\right)_{01} &= \tfrac{4}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n^{\mathbb{S}}} \tfrac{2\pi\lambda_1}{\sqrt{n}} ss(\lambda, y)cs(\lambda, y) = \left(\Sigma_{x,n}(u)\right)_{10} \\ \left(\Sigma_{x,n}(u)\right)_{02} &= \tfrac{4}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n^{\mathbb{S}}} \tfrac{2\pi\lambda_2}{\sqrt{n}} ss(\lambda, y)sc(\lambda, y) = \left(\Sigma_{x,n}(u)\right)_{20} \\ \left(\Sigma_{x,n}(u)\right)_{11} &= \tfrac{4}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n^{\mathbb{S}}} \tfrac{4\pi^2\lambda_1^2}{n} cs(\lambda, y)^2 \\ \left(\Sigma_{x,n}(u)\right)_{12} &= \tfrac{4}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n^{\mathbb{S}}} \tfrac{4\pi^2\lambda_1\lambda_2}{n} cs(\lambda, y)sc(\lambda, y) = \left(\Sigma_{x,n}(u)\right)_{21} \\ \left(\Sigma_{x,n}(u)\right)_{22} &= \tfrac{4}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n^{\mathbb{S}}} \tfrac{4\pi^2\lambda_2^2}{n} sc(\lambda, y)^2. \end{aligned}$$

Using a further shorthand[2]

$$\sigma_{ij} = (\sigma_{ij})_{x,n}(u) := \left(\Sigma_{x,n}(u)\right)_{ij}, \tag{3.18}$$

the determinant of $\Sigma_{x,n}(u)$ reads:

$$\det\left(\Sigma_{x,n}(u)\right) = \sigma_{00}\sigma_{11}\sigma_{22} - \sigma_{00}\sigma_{12}^2 - \sigma_{01}^2\sigma_{22} - \sigma_{02}^2\sigma_{11} + 2\sigma_{01}\sigma_{12}\sigma_{02}.$$

STEP 1: Method of proof.
We first claim that, uniformly w.r.t. u:

$$\int_{\mathbb{S}} \left(\sigma_{00}\sigma_{11}\sigma_{22} - 4\pi^4\right)^2 dx = O\left(\frac{1}{\mathcal{N}_n}\right), \tag{3.19}$$

[2]Careful too, to bear in mind the dependence of σ_{ij} on y and n.

and then that

$$\int_{\mathbb{S}}\int_{B(R)} \left(\det \Sigma_{x,n}(u) - \sigma_{00}\sigma_{11}\sigma_{22}\right)^2 \, dudx = O\left(\frac{1}{\mathcal{N}_n}\right), \tag{3.20}$$

upon which the triangle inequality yields (3.8).

STEP 2: Proof of Claim (3.19).
First, we establish the dominant behaviour:

$$\int_{\mathbb{S}} \sigma_{00}\sigma_{11}\sigma_{22}\, dx = 4\pi^4 + O\left(\frac{1}{\mathcal{N}_n}\right). \tag{3.21}$$

The integrand is

$$\sigma_{00}\sigma_{11}\sigma_{22} = \left(\frac{4}{\mathcal{N}_n}\right)^3 \sum_{\lambda,l,\mu} \frac{16\pi^4 l_1^2 \mu_2^2}{n^2} ss(\lambda,y)^2 cs(l,y)^2 sc(\mu,y)^2.$$

Integration w.r.t. x may be factored into 1-dimensional integrals; for each of these we can change the variable of integration to y_j, with $dy_j = dx_j$ and domain of integration $\left[\frac{2u_j}{\sqrt{n}}, 1+\frac{2u_j}{\sqrt{n}}\right]$, and thus we can separately analyse

$$\int_{\frac{2u_1}{\sqrt{n}}}^{1+\frac{2u_1}{\sqrt{n}}} \sin(\pi\lambda_1 y_1)^2 \cos(\pi l_1 y_1)^2 \sin(\pi\mu_1 y_1)^2 dy_1, \tag{3.22}$$

$$\int_{\frac{2u_2}{\sqrt{n}}}^{1+\frac{2u_2}{\sqrt{n}}} \sin(\pi\lambda_2 y_2)^2 \sin(\pi l_2 y_2)^2 \cos(\pi\mu_2 y_2)^2 dy_2. \tag{3.23}$$

To exhibit cancellation we use the double angle formulae

$$\cos A \cos B = \tfrac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$$
$$\sin A \sin B = \tfrac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

so that the *integrand* of (3.22) reads

$$\begin{aligned} &\tfrac{1}{8}(1-\cos 2\pi\lambda_1 y_1)(1+\cos 2\pi l_1 y_1)\,(1-\cos 2\pi\mu_1 y_1) \\ &\qquad = \tfrac{1}{8}\sum_{e\in\{0,1\}^3} (-\cos 2\pi\lambda_1 y_1)^{e_1} (\cos 2\pi l_1 y_1)^{e_2} (-\cos 2\pi\mu_1 y_1)^{e_3} \\ &\qquad = \tfrac{1}{8}\sum_{e\in\{0,1\}^3} (-1)^{e_1+e_3} \prod_{j=1}^{3} (\cos\left(\phi_j(y_1)\right))^{e_j}, \end{aligned} \tag{3.24}$$

where we have written $\phi_1(y_1) = 2\pi\lambda_1 y_1$, $\phi_2(y_1) = 2\pi l_1 y_1$ and $\phi_3(y_1) = 2\pi\mu_1 y_1$.

We can integrate trigonometric products of the type on the r.h.s. of (3.24) using a simple combinatoric formula, valid for $\phi_k(t) = 2\pi m_k t$ with $m_k \in \mathbb{Z}$:

$$\int_0^1 \prod_{k=1}^{K} \cos\left(\phi_k(t)\right)\, dt = \tfrac{1}{2^K} \sum_{e\in\{-1,1\}^K} \mathbf{1}_{\{e_1 m_1 + \ldots + e_K m_K = 0\}}.$$

Putting this to work on (3.22):

$$\begin{aligned}
&\int_{\frac{2u_1}{\sqrt{n}}}^{1+\frac{2u_1}{\sqrt{n}}} \sin(\pi\lambda_1 y_1)^2 \cos(\pi l_1 y_1)^2 \sin(\pi\mu_1 y_1)^2 dy_1 \\
&\qquad = \tfrac{1}{8} \sum_{e\in\{0,1\}^3} (-1)^{e_1+e_3} \int_0^1 \prod_{j=1}^{3} (\cos\left(\phi_j(t)\right))^{e_j} dt \\
&\qquad = \tfrac{1}{8} \sum_{e\in\{0,1\}^3} (-1)^{e_1+e_3} \left(\tfrac{1}{2^3} \sum_{a\in\{-1,1\}^3} \mathbf{1}_{\{a_1 e_1\lambda_1 + a_2 e_2 l_1 + a_3 e_3 \mu_1 = 0\}} \right) \\
&\qquad = \tfrac{1}{8} + \tfrac{1}{8} \sum_{\substack{e\in\{0,\pm1\}^3 \\ e\neq 0}} (-1)^{e_1+e_3} \left(\tfrac{1}{2^3} \sum_{a\in\{-1,1\}^3} \mathbf{1}_{\{a_1 e_1\lambda_1 + a_2 e_2 l_1 + a_3 e_3 \mu_1 = 0\}} \right) \\
&\qquad = \tfrac{1}{8} + O\left(\sum_{c\in\{0,\pm1\}^3\setminus 0} \mathbf{1}_{\{c_1\lambda_1 + c_2 l_1 + c_3\mu_1 = 0\}} \right), \qquad (3.25)
\end{aligned}$$

and similarly for (3.23) but with (λ_2, l_2, μ_2) in place of (λ_1, l_1, μ_1). Note the non-degenerate linear combination of lattice point ordinates in the indicator summands of (3.25). It is only when summing w.r.t. the lattice points λ, l, μ that we can see the cancellation within the big-O term:

$$\begin{aligned}
\int_{\mathbb{S}} \sigma_{00}\sigma_{11}\sigma_{22}\, dx &= \left(\frac{4}{\mathcal{N}_n}\right)^3 \sum_{\lambda,l,\mu} \frac{16\pi^4 l_1^2 \mu_2^2}{n^2} \left(\tfrac{1}{64} + O\left(\sum_{\substack{c,d\in\{0,\pm1\}^3 \\ (c,d)\neq(0,0)}} \mathbf{1}_{\substack{\{c_1\lambda_1 + c_2 l_1 + c_3\mu_1 = 0\} \\ \{d_1\lambda_2 + d_2 l_2 + d_3\mu_2 = 0\}}} \right) \right) \\
&= 4\pi^4 + O\left(\left(\frac{1}{\mathcal{N}_n}\right)^3 \sum_{\substack{c,d\in\{0,\pm1\}^3 \\ (c,d)\neq(0,0)}} \sum_{\lambda,l,\mu} \frac{\pi^4 l_1^2 \mu_2^2}{n^2} \mathbf{1}_{\substack{\{c_1\lambda_1 + c_2 l_1 + c_3\mu_1 = 0\} \\ \{d_1\lambda_2 + d_2 l_2 + d_3\mu_2 = 0\}}} \right)
\end{aligned}$$

Let (c, d) be any choice pair which is not $(0, 0)$. Then, w.l.o.g. one of c and d has a non-zero component, say, $c_1 \neq 0$. Thus we can rearrange so that

$$\lambda_1 = -\tfrac{c_2}{c_1} l_1 - \tfrac{c_3}{c_1}\mu_1,$$

and as a result the indicator is zero unless λ is specified by l and μ (or $\lambda_1 = 0$, but there is no such $\lambda \in \Lambda_n^{\mathbb{S}}$). Hence the sum over λ collapses:

$$\sum_{\lambda}\sum_{l}\sum_{\mu} \frac{\pi^4 l_1^2 \mu_2^2}{n^2}\, \mathbf{1}_{\{\lambda_1 = e_1^{-1}(e_2 l_1 + e_3 \mu_1)\}} \leq \sum_{l}\sum_{\mu} \frac{\pi^4 l_1^2 \mu_2^2}{n^2} \ll \mathcal{N}_n^2. \tag{3.26}$$

Since there are a bounded number ($= 27^2 - 1$) of such pairs (c, d), we have established the claim (3.21):

$$\int_{\mathbb{S}} \sigma_{00}\sigma_{11}\sigma_{22}\, dx = 4\pi^4 + O\left(\frac{1}{\mathcal{N}_n}\right).$$

We are now ready to consider (3.19). Expanding the integrand, the only term we have not yet computed is

$$\int_{\mathbb{S}} \left(\sigma_{00}\sigma_{11}\sigma_{22}\right)^2 dx. \tag{3.27}$$

However

$$\left(\sigma_{00}\sigma_{11}\sigma_{22}\right)^2 = \frac{4^{10}}{\mathcal{N}_n^6} \sum_{\substack{\lambda,l,\mu \\ m,\nu,\gamma}} \frac{l_1^2\mu_2^2\nu_1^2\gamma_2^2}{\pi^{-8} n^4} ss(\lambda, y)^2 cs(l, y)^2 sc(\mu, y)^2 ss(m, y)^2 cs(\nu, y)^2 sc(\gamma, y)^2,$$

and precisely the same treatment as used previously applies: after integration and disregard of any terms which include a non-degenerate linear combination indicator, the leading order term is seen to be

$$\begin{aligned} \frac{4^{10}}{\mathcal{N}_n^6} \sum_{\substack{\lambda,l,\mu \\ m,\nu,\gamma}} \frac{\pi^8 l_1^2\mu_2^2\nu_1^2\gamma_2^2}{2^{12} n^4} &= \frac{2^8\pi^8}{n^4\mathcal{N}_n^6} \sum_{\substack{\lambda,l,\mu \\ m,\nu,\gamma}} l_1^2\mu_2^2\nu_1^2\gamma_2^2 \\ &= \frac{2^8\pi^8}{n^4\mathcal{N}_n^6} \left(n\mathcal{N}_n/2\right)^4 \mathcal{N}_n^2 = 16\pi^8. \end{aligned}$$

Thus

$$\int_{\mathbb{S}} \left(\sigma_{00}\sigma_{11}\sigma_{22}\right)^2 dx = 16\pi^8 + O\left(\frac{1}{\mathcal{N}_n}\right),$$

and together with (3.21) this gives

$$\int_{\mathbb{S}} \left(\sigma_{00}\sigma_{11}\sigma_{22} - 4\pi^4\right)^2 dx = O\left(\frac{1}{\mathcal{N}_n}\right)$$

uniformly w.r.t. the argument u. (Cf. (3.18) for the dependency of σ_{ij} on u.)

STEP 3: Proof of claim (3.20).

$$\det\left(\Sigma_{x,n}(u)\right) - \sigma_{00}\sigma_{11}\sigma_{22} = -\sigma_{00}\sigma_{12}^2 - \sigma_{01}^2\sigma_{22} - \sigma_{02}^2\sigma_{11} + 2\sigma_{01}\sigma_{12}\sigma_{02},$$

and we claim that

$$\int_{\mathbb{S}} \left(-\sigma_{00}\sigma_{12}^2 - \sigma_{01}^2\sigma_{22} - \sigma_{02}^2\sigma_{11} + 2\sigma_{01}\sigma_{12}\sigma_{02}\right)^2 dx \ll O\left(\frac{1}{\mathcal{N}_n}\right),$$

expecting cancellation to come from the integral of *each* of the integrands individually. There are two patterns in the indices, one coming from the Leibniz determinant rule's 2-cycles, the other from 3-cycles. Importantly, each summand contains a factor of σ_{ij} with indices $i \neq j$. And each such factor features a non-square internal summand factor $\sin(\pi\lambda_j y_j)\cos(\pi\lambda_j y_j) = \frac{1}{2}\sin(2\pi\lambda_j y_j)$. This means every term has a non-identically-zero indicator for a linear combination of lattice point coordinates, resulting in cancellation up to a factor of at least $1/\mathcal{N}_n$, as we saw previously in (3.26), and we immediately see that

$$\int_{\mathbb{S}} \left(-\sigma_{00}\sigma_{12}^2 - \sigma_{01}^2\sigma_{22} - \sigma_{02}^2\sigma_{11} + 2\sigma_{01}\sigma_{12}\sigma_{02}\right) dx \ll O\left(\frac{1}{\mathcal{N}_n}\right). \tag{3.28}$$

Precisely the same discussion as for (3.27) (upon omission of the dominant term) applies to the integral of the square of (3.28), which must similarly be of order $O\left(\mathcal{N}_n^{-1}\right)$. This entire discussion being uniform w.r.t u, upon integration over $B(R)$ we conclude (3.20) and thus also (3.8). □

2.2. Simultaneous satisfaction: the sets $S_{R,n}$. It is not enough that the properties (3.9) and (3.10) of Corollary 3.2 hold independently of one another. We will need that they hold simultaneously on a family of sets which fill $\mathbb{S}$ in measure. The following lines are no more than a technical expression of this, taking care with our parameters.

LEMMA 3.3. *Suppose $(n_j)_{j\in\mathbb{N}}$ is any sequence of natural numbers such that we have the weak convergence $\mu_{n_j} \xrightarrow{*} \nu$. Then there exists a family (S_{R,n_j}) of subsets of $\mathbb{S}$ such that, for each $\eta > 0$, for any (large) $R > 0$ and any (small) $\rho, \kappa > 0$, there exists $\tilde{n}$ such that whenever $n_j \geq \tilde{n}$ we have simultaneous satisfaction of*

(i) $\operatorname{meas} S_{R,n_j} > 1 - \eta$
(ii) $\sup_{x\in S_{R,n_j}}\{\|K_{x,n_j} - K^{\dagger}_{x,n_j}\|_R^{\infty}\} \leq \rho$
(iii) $\inf_{x\in S_{R,n_j}}\{\inf_{u\in B(R)}\{\det \Sigma_{x,n_j}(u)\}\} \geq \kappa.$

PROOF. For notational convenience, in the proof we suppress the subscript j. Combining Markov's inequality with (3.9) we have that:

$$\text{meas}\{x \in \mathbb{S} : \|K_{x,n} - K_{x,n}^{\dagger}\|_R^{\infty} > \rho\} \leq \frac{1}{\rho}\int_{\mathbb{S}} \|K_{x,n} - K_{x,n}^{\dagger}\|_R^{\infty} dx \ll \frac{R}{\rho \mathcal{N}_n^{\frac{1}{7}}}$$

so provided we choose n such that $\mathcal{N}_n \gg \left(\frac{R}{\rho\eta}\right)^7$ we have existence of a set satisfying (i) and (ii) with $\eta/2$.

Next:

$$\text{meas}\left\{x \in \mathbb{S} : \left\|4\pi^4 - \det \Sigma_{x,n}\right\|_{B(R)}^{\infty} > \kappa\right\} \leq \tfrac{1}{\kappa}\int_{\mathbb{S}} \left\|4\pi^4 - \det \Sigma_{x,n}\right\|_{B(R)}^{\infty} dx.$$

If $x \in \mathbb{S}$ is any fixed point for which

$$\left\|4\pi^4 - \det \Sigma_{x,n}\right\|_{B(R)}^{\infty} := \sup_{u \in B(R)} \left|4\pi^4 - \det \Sigma_{x,n}(u)\right| \leq \kappa,$$

then

$$\inf_{u \in B(R)} \det \Sigma_{x,n}(u) \geq 4\pi^4 - \kappa.$$

Thus

$$\text{meas}\left\{x \in \mathbb{S} : \inf_{u \in B(R)} \det \Sigma_{x,n}(u) \geq 4\pi^4 - \kappa\right\} \geq 1 - \tfrac{1}{\kappa}\int_{\mathbb{S}} \left\|4\pi^4 - \det \Sigma_{x,n}\right\|_{B(R)}^{\infty} dx$$

and, provided we fix $\kappa \in (0, 4\pi^4)$, this inequality has content. Choosing $\kappa = 2\pi^4$ and using (3.10) gives

$$\text{meas}\left\{x \in \mathbb{S} : \inf_{u \in B(R)} \det \Sigma_{x,n}(u) \geq 2\pi^4\right\} \geq 1 - O\left(\frac{R}{\mathcal{N}_n^{\frac{1}{5}}}\right),$$

so provided we choose n such that $\mathcal{N}_n \gg \left(\frac{R}{\eta}\right)^5$ we have existence of a set satisfying (i) and (iii) with $\eta/2$.

Now if $A, B \subset \mathbb{S}$, with $\text{meas}\, A > 1 - \eta/2$ and $\text{meas}\, B > 1 - \eta/2$, then

$$\text{meas}\,(A \cap B) \geq 1 - \eta; \tag{3.29}$$

and so provided that n is such that

$$\mathcal{N}_n \geq \max\left\{\left(\frac{R}{\eta}\right)^5, \left(\frac{R}{\rho\eta}\right)^7\right\},$$

we simultaneously satisfy (i), (ii) and (iii). Since $\mathcal{N}_n \to \infty$ with the hypothesised n, we are done. □

3. Local limiting mean nodal intensity

We will now use Lemma 3.3, which encodes the statement that: on a large proportion of $\mathbb{S}$, with high probability, boundary-adapted arithmetic random waves simultaneously have desirable non-degeneracy and stationarity properties for large values of the scaling parameter. The non-degeneracy property, together with a.s. smoothness, enable us to treat with the values of the random function on nets and interpolate between the points of the net. The limiting stationarity will enable us to reduce to the limiting mean nodal intensity result for translation-invariant Gaussian functions that we saw as Theorem 2.10. The upshot is a local limiting mean nodal intensity lemma, Lemma 3.4, which we introduce in Section 3.1 and prove in Section 3.2.

3.1. Statement of Lemma 3.4.

For $x \in \mathbb{S}$ we define the event

$$\Omega_{x,n,R}(\epsilon) := \left\{ \left| \frac{N(x, R/\sqrt{n}; f_n)}{\operatorname{vol} B(R)} - a_\nu \right| > \epsilon \right\}, \tag{3.30}$$

where, recall, $N(x, R/\sqrt{n}; f_n)$ denotes, equivalently:

(i) the number of nodal components of f_n which are strictly contained within the open ball $B(x, R/\sqrt{n})$

(ii) the number of nodal components of the scaled random function $f_{x,n}$ which are strictly contained within the open ball $B(R)$,

and where the constant a_ν is the limiting constant guaranteed by Theorem 2.10 for the translation-invariant Gaussian function whose spectral measure is ν.

The key step in proving Theorem 1.1 is to show that under suitable conditions on the parameters n and R, the probability of the event $\Omega_{x,n,R}(\epsilon)$ vanishes (in a specific limit involving R and n) uniformly w.r.t. x for a sequence of subsets that fill an arbitrarily large proportion of the measure of $\mathbb{S}$. The family of sets we will need are the $(S_{R,n})$ from Lemma 3.3. Now we state the lemma which shows that, for this family, we have a local limiting nodal intensity.

LEMMA 3.4. *Let (f_n) be the ensemble of boundary-adapted arithmetic random waves and suppose that $(n_j)_{j\in\mathbb{N}}$ is a sequence of natural numbers such that $\mu_{n_j} \xrightarrow{*} \nu$ for a non-atomic limiting measure ν. Fix $\epsilon > 0$ and $\eta > 0$. Then*

$$\lim_{R\to\infty} \overline{\lim_{j\to\infty}} \sup_{x\in S_{R,n_j}} \mathbb{P}\left\{\Omega_{x,n_j,R}(\epsilon)\right\} = 0.$$

REMARK. For notational convenience we suppress the subscript j from n_j in the argument that follows, with the understanding that every occurrence of the

parameter n has a 'hidden j'. Where limits are taken, we write $n \to \infty$ rather than $j \to \infty$. Since the sequence (n_j) and target limiting measure ν are fixed in the statement of the lemma, no confusion should arise from this abuse of notation.

3.2. Proof of Lemma 3.4.

Preliminary reduction. Fix the parameter $R > 2$. Write F for the translation invariant Gaussian function with spectral measure ν, and a_ν for the corresponding mean nodal intensity constant guaranteed by Theorem 2.10. If we can show that for all $t \in \mathbb{R}$ the two inequalities

$$\overline{\lim_{n\to\infty}} \sup_{S_{R,n}} \mathbb{P}\left\{N(R; f_{x,n}) > t\right\} \leq \mathbb{P}\left\{N(R+1; F) > t\right\}, \tag{3.31}$$

$$\overline{\lim_{n\to\infty}} \sup_{S_{R,n}} \mathbb{P}\left\{N(R; f_{x,n}) < t\right\} \leq \mathbb{P}\left\{N(R-1; F) < t\right\}, \tag{3.32}$$

hold, then by setting $t = (a_\nu \pm \epsilon) \operatorname{vol} B(R)$ we obtain

$$\overline{\lim_{n\to\infty}} \sup_{S_{R,n}} \mathbb{P}\left\{\left|\frac{N(R; f_{x,n})}{\operatorname{vol} B(R)} - a_\nu\right| > \epsilon\right\} \leq \mathbb{P}\left\{\frac{N(R+1; F)}{\operatorname{vol} B(R)} - a_\nu > \epsilon\right\} + \mathbb{P}\left\{\frac{N(R-1; F)}{\operatorname{vol} B(R)} - a_\nu < -\epsilon\right\}.$$

Appealing to Theorem 2.10, we have that both terms on the r.h.s. of the inequality vanish as $R \to \infty$, and by squeezing the lemma is proved. □

Proof of inequality (3.31). Fix $R > 2$ throughout; let $\delta > 0$; let large $A > 0$ and small $a > 0$ be parameters to be chosen and define the set of functions

$$E(A, a) = \left\{g \in C^1(B^+_{R+1}) : \|g\|_{C^1(\overline{B}_{R+1}),1} \leq A, \ \& \min_{x\in\overline{B}_{R+1}} \max\left\{|g(x)|, |\nabla g(x)|\right\} \geq a\right\}.$$

Using Corollary 2.16 for $x \in S_{R,n}$, we can choose A and a, so that for all sufficiently large n we have

$$\mathbb{P}\{f_{x,n} \notin E(A, a)\} = \mathbb{P}\left(\Omega^A_a(f_{x,n})\right) < \delta,$$

and also

$$\mathbb{P}\{F \notin E(A, a)\} < \delta,$$

the latter following from Theorem 2.10, conditions $(\rho 1)$ – $(\rho 4)$.

Now fix a finite $a/(2A)$-*net* in $\overline{B}(R+1)$, which we denote by X. This is a set of points so that, for any $y \in \overline{B}(R+1)$, we have

$$B_y(\tfrac{a}{4A}) \cap X \neq \varnothing.$$

The *trace* on X of a function g, is the vector $(g(x))_{x\in X} \in \mathbb{R}^{|X|}$. For sufficiently smooth functions, treating with the trace is a useful simplification: if $g, h \in E(A, a)$ and $\sup_{x\in X} |g(x) - h(x)| \leq a/2$, then $|g(u) - h(u)| \leq a$ for all $u \in B(R + 1)$.

Let $E \subset \mathbb{R}$ be the set of traces on X of functions $g \in E(A, a)$ which have the additional property that $N(R; g) > t$. The fact that $\|g\|_{C^1(\overline{B}_{R+1}),1} \leq A$, guarantees that $E \subset [-A, A]^{|X|} \subset \mathbb{R}^{|X|}$ is bounded, and so permits us to fix a bump function $\varphi \in C_0^\infty\left(\mathbb{R}^{|X|}\right)$ such that

$$\varphi(\xi) = \begin{cases} 1 & \xi \in E, \\ 0 & \xi \notin E_{+a/2}. \end{cases}$$

Suppose that $\xi \in \mathbb{R}^{|X|}$ is a Gaussian random vector. Then for some $0 < \alpha < 1$,

$$\mathbb{E}\{\varphi(\xi)\} = \int_\Omega 0 \times \left(1_{\left(\mathbb{R}^{|X|}\setminus E_{+a/2}\right)}(\xi)\right) d\mathbb{P} + \int_\Omega \alpha \times \left(1_{\left(E_{+a/2}\setminus E\right)}(\xi)\right) d\mathbb{P} + \int_\Omega 1_E(\xi) d\mathbb{P},$$

and in particular

$$\mathbb{E}\{\varphi(\xi)\} = 0 \times \mathbb{P}\{\varphi = 0\} + \alpha \times \mathbb{P}\{0 < \varphi < 1\} + 1 \times \mathbb{P}\{\varphi = 1\}. \tag{3.33}$$

By replacing α in the equality (3.33) with its infinum $\alpha = 0$ and its supremum $\alpha = 1$ respectively, we deduce the useful inequality

$$\mathbb{P}\{\varphi(\xi) = 1\} \leq \mathbb{E}\{\varphi(\xi)\} \leq \mathbb{P}\{\varphi(\xi) > 0\}. \tag{3.34}$$

By $f_{x,n}|_X$, we denote the vector of trace values of $f_{x,n}$ on the net X. We are now ready to consider and manipulate the set of events $\{\omega \in \Omega_n : N(R; f_{x,n}^{(\omega)}) > t\}$, suppressing the ω for notational convenience.

$$\begin{aligned} \{N(R; f_{x,n}) > t\} &\subset \{f_{x,n} \in E(A, a) \;\&\; N(R; f_{x,n}) > t\} \cup \{f_{x,n} \notin E(A, a)\} \\ &\subset \{f_{x,n} \in E(A, a) \;\&\; f_{x,n}|_X \in E^{>}\} \cup \{f_{x,n} \notin E(A, a)\} \\ &\subset \{\varphi\left(f_{x,n}|_X\right) = 1\} \cup \{f_{x,n} \notin E(A, a)\}. \end{aligned} \tag{3.35}$$

Taking the $\mathbb{P}$-measure of each side of (3.35) and using the lower half of (3.34) with $\xi = f_{x,n}|_X$:

$$\begin{aligned} \mathbb{P}\{N(R; f_{x,n}) > t\} &\leq \mathbb{P}\{\varphi(f_{x,n}|_X) = 1\} + \mathbb{P}\{f_{x,n} \notin E(A, a)\} \\ &< \mathbb{E}\{\varphi(f_{x,n}|_X)\} + \delta. \end{aligned} \tag{3.36}$$

Next, we couple the probability spaces, by proving that, taking n sufficiently large we can guarantee that

$$\mathbb{E}\{\varphi(f_{x,n}|_X)\} \leq \mathbb{E}\{\varphi(F|_X)\} + \delta. \tag{3.37}$$

To prove (3.37), we first use Fourier inversion on the bump function:

$$\begin{aligned}\mathbb{E}\{\varphi(f_{x,n}|_X)\} &= \mathbb{E}\left\{\int_{\mathbb{R}^{|X|}} \hat{\varphi}(\lambda) e^{2\pi i \lambda \cdot f_{x,n}|_X} d\lambda\right\} \\ &= \int_{\mathbb{R}^{|X|}} \hat{\varphi}(\lambda) \mathbb{E}\left\{e^{2\pi i \lambda \cdot f_{x,n}|_X}\right\} d\lambda = \int_{\mathbb{R}^{|X|}} \hat{\varphi}(\lambda) e^{-\pi \left(K_{x,n}^X\right)\lambda\cdot\lambda} d\lambda, \end{aligned} \tag{3.38}$$

where we write $\left(K_{x,n}^X\right)$ to denote the covariance *matrix* of the random trace vector $f_{x,n}|_X$. Recall that the net X has cardinality $|X| \asymp \frac{A^2R^2}{a^2}$ and so the matrix $\left(K_{x,n}^X\right)$ has $\asymp A^4R^4a^{-4}$ entries.

By the same argument used in (3.38) we have the expression

$$\begin{aligned}\mathbb{E}\left\{\varphi(F|_X)\right\} &= \mathbb{E}\left\{\int_{\mathbb{R}^{|X|}} \hat{\varphi}(\lambda) e^{2\pi i \lambda \cdot F|_X} d\lambda\right\} \\ &= \int_{\mathbb{R}^{|X|}} \hat{\varphi}(\lambda) \mathbb{E}\left\{e^{2\pi i \lambda \cdot F|_X}\right\} d\lambda = \int_{\mathbb{R}^{|X|}} \hat{\varphi}(\lambda) e^{-\pi \left(k_\nu^X\right)\lambda\cdot\lambda} d\lambda \end{aligned} \tag{3.39}$$

where $\left(k_\nu^X\right)$ is the covariance matrix of the random trace vector $F|_X$.

To couple the expressions (3.38) and (3.39) we begin by using the limiting translation-invariance property quantified in Lemma 3.3. For whichever small $\rho = \rho(\delta)$ we might choose, there is an n_δ such that $n \geq n_\delta$ implies that the set of base-points

$$S_{R,\rho,n} = \{x \in \mathbb{S} : \|K_{x,n} - K_{x,n}^\dagger\|_R^\infty < \rho\} \tag{3.40}$$

has measure greater than $1 - \eta$. From here until the end, we assume that $x(= x_n)$ is a point taken arbitrarily from the set $S_{R,n}$. For x so-chosen, (3.40) states that we have convergence as $n \to \infty$ of $K_{x,n} \to K_{x,n}^\dagger$ uniformly on $B(R) \times B(R)$. Recalling that $K_{x,n}^\dagger$ is independent of the basepoint x, we have uniform convergence to the covariance function k_ν (cf. (3.5)), and hence also pointwise convergence as functions of λ:

$$\hat{\varphi}(\lambda) e^{-\pi \left(K_{x,n}^X\right)\lambda\cdot\lambda} \to \hat{\varphi}(\lambda) e^{-\pi \left(k_\nu^X\right)\lambda\cdot\lambda}.$$

Each of these functions are dominated by $\hat{\varphi}$, which is integrable on account of being the Fourier transform of a bump function. By the dominated convergence theorem

$$\int_{\mathbb{R}^{|X|}} \hat{\varphi}(\lambda) e^{-\pi \left(K_{x,n}^X\right)\lambda\cdot\lambda} d\lambda \to \int_{\mathbb{R}^{|X|}} \hat{\varphi}(\lambda) e^{-\pi \left(k_\nu^X\right)\lambda\cdot\lambda} d\lambda$$

as $n \to \infty$, and thus we can find n_δ such that $n \geq n_\delta$ ensures

$$\left| \int_{\mathbb{R}^{|X|}} \hat{\varphi}(\lambda) e^{-\pi \left(K_{x,n}^X\right)\lambda\cdot\lambda} d\lambda - \int_{\mathbb{R}^{|X|}} \hat{\varphi}(\lambda) e^{-\pi \left(k_\nu^X\right)\lambda\cdot\lambda} d\lambda \right| \leq \delta;$$

or equivalently

$$|\mathbb{E}\{\varphi(f_{x,n}|_X)\} - \mathbb{E}\{\varphi(F|_X)\}| \leq \delta,$$

which implies (3.37).

Using the upper half of (3.34) with $\xi = F|_X$:

$$\mathbb{E}\{\varphi(F|_X)\} \leq \mathbb{P}\{\varphi(F|_X) > 0\}.$$

We now do something similar to (3.35), but in reverse. Firstly,

$$\begin{aligned}\{\varphi(F|_X) > 0\} &\subset \{F|_X \in E_{+a/2}\} \\ &\subset \{F|_X \in E_{+a/2} \,\&\, F \in E(A,a)\} \cup \{F \notin E(A,a)\}.\end{aligned}$$

Now note that, if $F \in E(A,a)$ and $F|_X \in E_{+a/2}$, then there exists a function $g \in E(A,a)$ with trace $g|_X \in E$ and $N(R;g) > t$, and for which $|F-g| < a/2$ on X. This implies $|F-g| < a$ on the whole of $\overline{B}(R+1)$. Applying Lemma 2.14 we deduce that $N(R+1;F) \geq N(R;g)$, and so

$$\{\varphi(F|_X) > 0\} \subset \{N(R+1,F) > t\} \cup \{F \notin E(A,a)\}.$$

Taking $\mathbb{P}$-measure

$$\mathbb{P}\{\varphi(F|_X) > 0\} < \mathbb{P}\{N(R+1,F) > t\} + \delta. \tag{3.41}$$

Combining the steps (3.36), (3.37) and (3.41), we have that for all n sufficiently large

$$\mathbb{P}\{N(R;f_{x,n}) > t\} < \mathbb{P}\{N(R+1,F) > t\} + 3\delta,$$

which together with the uniformity across $x \in S_{R,n}$ implies (3.31): that

$$\overline{\lim_{n\to\infty}} \sup_{S_{R,n}} \mathbb{P}\{N(R;f_{x,n}) > t\} \leq \mathbb{P}\{N(R+1,F) > t\}.$$

□

Proof of inequality (3.32). We need make only superficial changes to the method above:

- The set of traces E is instead defined as the set of traces of functions in $h \in E(A,a)$ with the additional property that $N(R;h) < t$. This redefines the bump function φ, but all of the subsequent steps are still valid up until the point when we are constructing a superset for $\{\varphi(F|_X) > 0\}$.
- Here we argue instead that if $F \in E(A,a)$ and $F|_X \in E_{+a/2}$, then there exists a function $h \in E(A,a)$ with trace $h|_X \in E$ and $N(R;h) < t$, and for

which $|F-h|<a/2$ on X. Again this implies $|F-h|<a$ on the whole of $\overline{B}(R+1)$, but here we deduce that $N(R-1;F)\leq N(R;g)$, and so

$$\{\varphi\,(F|_X)>0\}\subset\{N(R-1,F)<t\}\cup\{F\notin E(A,a)\},$$

from which it follows that

$$\mathbb{P}\{\varphi\,(F|_X)>0\}<\mathbb{P}\{N(R-1,F)<t\}+\delta. \tag{3.42}$$

Combining the analogies of steps (3.36) and (3.37) with (3.42), we conclude (3.32). □

CHAPTER 4

Main Theorems

In this chapter we tie together results from Chapters 2 and 3 to prove the main theorems of this thesis, Theorems 1.1 and 1.2.

1. Limiting mean nodal intensity

The method of proof for Theorem 1.1 follows and builds upon that of Nazarov and Sodin [**18**] described in Chapter 2, with an important addition in the form of Lemma 3.4. This lemma, we recall, demonstrates that under suitable conditions the event that the number of nodal components in a small ball around a given point $x \in \mathbb{S}$ is significantly different from that expected, is of vanishing probability. This ensures that probability concentrates on events for which the nodal intensity is very close to a special positive constant a_ν.

1.1. Proof of Theorem 1.1.

1.1.1. *Outline.* The first step is to categorise nodal components according to their size. Components of very small volume do not occur: the Faber–Krahn inequality, Lemma 2.2, gives a deterministic lower bound on the volume which can be contained in a nodal domain in terms of the eigenvalue of the Laplace eigenfunction, with a uniform constant δ (implicit in (2.1)). More precisely, if G is the nodal domain of f_n for any nodal component $\gamma = \partial G$, then $\mathrm{vol}(G) \geq \delta \frac{1}{n}$.

With this in mind, we will categorise the nodal components as *normal* or as *long*. Because Lemma 3.4 only sees the (blown-up) components which fit strictly within the ball of radius R, we need to be able to discard the contribution of components of diameter larger than this by a different argument. Lemma 3.4 cannot be applied to the entirety of $\mathbb{S}$: for each value of the parameter n we will need to avoid a set of positive measure near to points at which the distribution of f_n degenerates. Here we will be able to use the Faber–Krahn inequality to give an upper bound for the number of nodal components that could possible be present.

Normal components will have diameter comparable with $1/\sqrt{n}$ and, importantly, constitute the bulk of the nodal count (for large values of the parameter n). Lemma 3.4 effectively captures and counts these normal components, thereby

describing the asymptotic behaviour of $N(f_{n_j})$ as $j \to \infty$. We now make the discussion in this outline rigorous.

1.1.2. *Categorising components by size.* We start by providing a precise definition of the size categories of nodal components:

DEFINITION. Let D be a positive real parameter. A connected component $\gamma \in \mathcal{Z}(f_n)$ is called:

(i) *D–long*, if $\operatorname{diam}(\gamma) > D/\sqrt{n}$.
(ii) *D–normal* (or for convenience, simply *normal*), otherwise.

We denote the number of D–long components of f_n by $N_{D-long}(f_n)$, and denote the number of normal components by $N_{norm}(f_n)$.

In light of this categorisation and the preceding discussion, proving Theorem 1.1 amounts to writing

$$\mathbb{E}\left|\frac{N(f_{n_j})}{n_j} - a_\nu\right| \leq \mathbb{E}\left|\frac{N_{norm}(f_{n_j})}{n_j} - a_\nu\right| + \mathbb{E}\left|\frac{N_{D-long}(f_{n_j})}{n_j}\right|, \tag{4.1}$$

then showing that the r.h.s. vanishes as $j \to \infty$. In the remainder of Section 1 we show that the two r.h.s. summands of (4.1) each vanish in the high-energy limit.

1.1.3. *Discarding D–long components.* Here we make use of the upper bound of Donnelly–Fefferman in Lemma 2.4, which states that $\mathcal{L}(Z(f_n)) \ll \sqrt{n}$. Since each $D-$long component γ satisfies

$$\mathcal{L}(\gamma) > \operatorname{diam}(\gamma) > D/\sqrt{n},$$

we attain the upper bound

$$N_{D-long}(f_n) \ll \frac{\mathcal{L}(Z(f_n))}{D/\sqrt{n}} \ll \frac{n}{D}. \tag{4.2}$$

Provided we are permitted to take the parameter $D \to \infty$, the r.h.s. of (4.2) vanishes for each sample of f_n, thus the corresponding expectation on the r.h.s. of (4.1) also vanishes.

1.1.4. *Counting normal components.* To show that

$$\mathbb{E}\left|\frac{N_{norm}(f_{n_j})}{n_j} - a_\nu\right| \to 0 \qquad j \to \infty,$$

we begin with the (deterministic) integral-geometric sandwich, Lemma 2.13. Recalling the notation there, and applying it with $N = N_{norm}$ (so that for instance:

$N_{norm}(x, r; f_n)$ denotes the number of *normal* nodal components strictly contained within the open ball $B(x, r)$), we have that for all $r > 0$:

$$\int_{\mathbb{S}} \frac{N_{norm}(x, r; f_n)}{\operatorname{vol} B(r)} dx \leq N_{norm}(f_n) \leq \int_{\mathbb{S}} \frac{N^*_{norm}(x, r; f_n)}{\operatorname{vol} B(r)} dx. \tag{4.3}$$

Since all normal components γ have $\operatorname{diam}(\gamma) < D/\sqrt{n}$ we have

$$N^*_{norm}(x, r; f_n) \leq N_{norm}(x, r + D/\sqrt{n}; f_n).$$

If we now fix a large R, choose parameters $r = R/\sqrt{n}$ and $D = \sqrt{R}$, and write $R^* = R + D$, then the sandwich (4.3) reads:

$$\int_{\mathbb{S}} \frac{N_{norm}(x, R/\sqrt{n}; f_n)}{\operatorname{vol} B(R/\sqrt{n})} dx \leq N_{norm}(f_n) \leq \int_{\mathbb{S}} \frac{N_{norm}(x, (R+D)/\sqrt{n}; f_n)}{\operatorname{vol} B(R/\sqrt{n})} dx. \tag{4.4}$$

Dividing through by $1/n$ and manipulating the r.h. inequality so as to have numerator and denominator measuring balls of comparable radii:

$$\int_{\mathbb{S}} \frac{N_{norm}(x, R/\sqrt{n}; f_n)}{\operatorname{vol} B(R)} dx \leq \frac{N_{norm}(f_n)}{n} \leq \left(1 + \tfrac{\sqrt{R}}{R}\right)^2 \int_{\mathbb{S}} \frac{N_{norm}(x, R^*/\sqrt{n}; f_n)}{\operatorname{vol} B(R^*)} dx.$$

Thus for any small parameter $\epsilon > 0$, taking R sufficiently large we have

$$\int_{\mathbb{S}} \frac{N_{norm}(x, R/\sqrt{n}; f_n)}{\operatorname{vol} B(R)} dx \leq \frac{N_{norm}(f_n)}{n} \leq (1 + \epsilon) \int_{\mathbb{S}} \frac{N_{norm}(x, R^*/\sqrt{n}; f_n)}{\operatorname{vol} B(R^*)} dx. \tag{4.5}$$

We now subtract the target nodal intensity a_ν (cf. (3.30)), from each part of (4.5) to give

$$A^- \leq \frac{N_{norm}(f_n)}{n} - a_\nu \leq A^+, \tag{4.6}$$

where

$$A^- := \int_{\mathbb{S}} \frac{N_{norm}(x, R/\sqrt{n}; f_n)}{\operatorname{vol} B(R)} dx - a_\nu, \tag{4.7}$$

and

$$A^+ := (1 + \epsilon) \int_{\mathbb{S}} \frac{N_{norm}(x, R^*/\sqrt{n}; f_n)}{\operatorname{vol} B(R^*)} dx - a_\nu. \tag{4.8}$$

From (4.6), we have

$$\left| \frac{N_{norm}(f_n)}{n} - a_\nu \right| \leq \max\left\{ \left|A^-\right|, \left|A^+\right| \right\} \leq \left|A^-\right| + \left|A^+\right|, \tag{4.9}$$

which holds almost surely, thus also in expectation:

$$\mathbb{E}\left|N_{norm}(f_n)/n - a_\nu\right| \leq \mathbb{E}\left|A^+\right| + \mathbb{E}\left|A^-\right|. \tag{4.10}$$

We begin with a preliminary reduction for bounding $\mathbb{E}\,|A^+|$, rewriting the (deterministic) equality (4.8) as

$$A^+ = \int_{\mathbb{S}} \left(\frac{N_{norm}(x, R^*/\sqrt{n}; f_n)}{\operatorname{vol} B(R^*)} - a_\nu \right) dx + \epsilon \int_{\mathbb{S}} \left(\frac{N_{norm}(x, R^*/\sqrt{n}; f_n)}{\operatorname{vol} B(R^*)} \right) dx,$$

to see that

$$\begin{aligned} \left|A^+\right| &\leq \left| \int_{\mathbb{S}} \left(\frac{N_{norm}(x, R^*/\sqrt{n}; f_n)}{\operatorname{vol} B(R^*)} - a_\nu \right) dx \right| + \epsilon \left| \int_{\mathbb{S}} \frac{N_{norm}(x, R^*/\sqrt{n}; f_n)}{\operatorname{vol} B(R^*)} dx \right| \\ &\leq \int_{\mathbb{S}} \left| \frac{N_{norm}(x, R^*/\sqrt{n}; f_n)}{\operatorname{vol} B(R^*)} - a_\nu \right| dx + \epsilon \int_{\mathbb{S}} \frac{N_{norm}(x, R^*/\sqrt{n}; f_n)}{\operatorname{vol} B(R^*)} dx. \end{aligned} \tag{4.11}$$

The second summand on the r.h.s. of (4.11) is almost surely $O(\epsilon)$, which can be quickly shown using the Faber–Krahn inequality, Lemma 2.2: every nodal domain contains volume greater than δ/n for some constant δ (implicit in (2.1)), and so

$$N_{norm}(x, R^*/\sqrt{n}; f_n) \leq \frac{\operatorname{vol} B(R^*/\sqrt{n})}{\delta/n} = \delta^{-1} \operatorname{vol} B(R^*). \tag{4.12}$$

Thus the *expectation* of the second summand is also $O(\epsilon)$.

With this additional term dealt with, all that remains is show that the expected value of the first summand of (4.11) is $O(\epsilon)$ too. This requires more delicate analysis, for which we will use Lemma 3.4. Precisely the same procedure for this first integral works for bounding the expected value of (4.7), and since the latter is notationally neater (but exactly the same upon exchanging R for R^*), we present that argument only.

Before deploying Lemma 3.4, we partition the domain of integration into sets[1] $S_{R,n}(\eta)$ whose existence was established in Lemma 3.3, and $\mathbb{S} \setminus S_{R,n}(\eta)$ its complement. Temporarily abusing notation by suppressing the integrand, the expectation of the first summand of (4.11) under this partition may be expressed as:

$$\int_\Omega \int_{\mathbb{S}} = \int_\Omega \left(\int_{\mathbb{S} \setminus S_{R,n}(\eta)} + \int_{S_{R,n}(\eta)} \right) = \int_\Omega \int_{\mathbb{S} \setminus S_{R,n}(\eta)} + \int_\Omega \int_{S_{R,n}(\eta)}. \tag{4.13}$$

Recall the definition (3.30) of the event $\Omega_{x,n,R}(\epsilon)$ and the assertion of Lemma 3.4:

$$\lim_{R \to \infty} \overline{\lim_{n \to \infty}} \sup_{S_{R,n}(\eta)} \mathbb{P}\left(\Omega_{x,n,R}(\epsilon)\right) = 0.$$

[1]We make explicit the small parameter $\eta > 0$ which was fixed (and thus notationally suppressed) within Lemma 3.4.

We can now begin to develop (4.13) so as to make use of this assertion, first reversing the order of integration, then partitioning the expectation according to $\Omega_{x,n,R}(\epsilon)$:

$$\begin{aligned}\int_\Omega \int_{\mathbb{S}} &= \int_\Omega \int_{\mathbb{S}\setminus S_{R,n}(\eta)} + \int_{S_{R,n}(\eta)} \left(\int_{\Omega\setminus\Omega_{x,n,R}(\epsilon)} + \int_{\Omega_{x,n,R}(\epsilon)} \right) \\ &= \int_\Omega \int_{\mathbb{S}\setminus S_{R,n}(\eta)} + \int_{S_{R,n}(\eta)} \int_{\Omega\setminus\Omega_{x,n,R}(\epsilon)} + \int_{S_{R,n}(\eta)} \int_{\Omega_{x,n,R}(\epsilon)} . \end{aligned} \tag{4.14}$$

For the first integral of (4.14):

$$\begin{aligned}\int_\Omega \int_{\mathbb{S}\setminus S_{R,n}(\eta)} &\left| \frac{N_{norm}(x, R/\sqrt{n}; f_n)}{\operatorname{vol} B(R)} - a_\nu \right| dx d\mathbb{P} \\ &\leq \int_\Omega \int_{\mathbb{S}\setminus S_{R,n}(\eta)} \left(\frac{N_{norm}(x, R/\sqrt{n}; f_n)}{\operatorname{vol} B(R)} + a_\nu \right) dx d\mathbb{P} \\ &\leq \int_\Omega \eta \left(\delta^{-1} + a_\nu \right) d\mathbb{P} \ll \eta, \end{aligned} \tag{4.15}$$

where the last inequality re-employs the bound (4.12).
For the second integral of (4.14) we know that the nodal intensity is within ϵ of the mean:

$$\begin{aligned}\int_{S_{R,n}(\eta)} \int_{\Omega\setminus\Omega_{x,n,R}(\epsilon)} &\left| \frac{N_{norm}(x, R/\sqrt{n}; f_n)}{\operatorname{vol} B(R)} - a_\nu \right| d\mathbb{P} dx \\ &\leq \int_{S_{R,n}(\eta)} \epsilon \mathbb{P}\left(\Omega \setminus \Omega_{x,n,R}(\epsilon)\right) dx \leq \epsilon(1-\eta). \end{aligned} \tag{4.16}$$

For the third integral of (4.14) we will be able to apply the main lemma, so that it will vanish when n and R are large:

$$\begin{aligned}\int_{S_{R,n}(\eta)} \int_{\Omega_{x,n,R}(\epsilon)} &\left| \frac{N_{norm}(x, R/\sqrt{n}; f_n)}{\operatorname{vol} B(R)} - a_\nu \right| d\mathbb{P} dx \\ &\leq \int_{S_{R,n}(\eta)} \left(\delta^{-1} + a_\nu \right) \mathbb{P}\left(\Omega_{x,n,R}(\epsilon)\right) dx \\ &\ll \sup_{S_{R,n}(\eta)} \mathbb{P}\left(\Omega_{x,n,R}(\epsilon)\right). \end{aligned} \tag{4.17}$$

Drawing together (4.15), (4.16) and (4.17), we see that

$$\mathbb{E}\left|A^-\right| \ll \sup_{S_{R,n}(\eta)} \mathbb{P}\left(\Omega_{x,n,R}(\epsilon)\right) + \eta + \epsilon(1-\eta),$$

and the same procedure with R replaced by $R^* = R + D$ shows that

$$\mathbb{E}\left|A^+\right| \ll \sup_{S_{R+D,n}(\eta)} \mathbb{P}\left(\Omega_{x,n,R+D}(\epsilon)\right) + \eta + \epsilon(1-\eta) + \epsilon.$$

Returning to (4.10) we have

$$\mathbb{E}\,|N_{norm}(f_n)/n - a_\nu| \ll \sup_{S_{R+D,n}(\eta)} \mathbb{P}\left(\Omega_{x,n,R+D}(\epsilon)\right) + \sup_{S_{R,n}(\eta)} \mathbb{P}\left(\Omega_{x,n,R}(\epsilon)\right) + \eta + \epsilon(1-\eta) + \epsilon. \tag{4.18}$$

1.1.5. *Concluding the proof.* We recall that the proof of the Theorem 1.1 rests upon showing that the r.h.s. of inequality (4.1) vanishes. Combining the estimates (4.2) and (4.18), we have

$$\begin{aligned}\mathbb{E}\left|\frac{N(f_{n_j})}{n_j} - a_\nu\right| &\leq \mathbb{E}\left|\frac{N_{norm}(f_{n_j})}{n_j} - a_\nu\right| + \mathbb{E}\left|\frac{N_{D-long}(f_{n_j})}{n_j}\right| \\ &\ll \sup_{S_{R+D,n}(\eta)} \mathbb{P}\left(\Omega_{x,n,R+D}(\epsilon)\right) + \sup_{S_{R,n}(\eta)} \mathbb{P}\left(\Omega_{x,n,R}(\epsilon)\right) + \eta - \epsilon\eta + \epsilon + D^{-1}. \end{aligned} \tag{4.19}$$

Since the l.h.s. of (4.19) does not depend on the parameters R, ϵ, η, all that remains of the proof of Theorem 1.1 is to justify a valid choice of these parameters for which (4.19) vanishes. Recall from (4.4) that we've chosen $D = \sqrt{R}$, which implies that the term D^{-1} vanishes as long as $R \to \infty$. To deal with those terms that remain, provided that we take limits in the order $n \to \infty$, $R \to \infty$, $\epsilon \to 0$, $\eta \to 0$, the result follows. Using linearity of limits we treat each r.h.s. summand of (4.19) separately, giving

$$\begin{aligned}&\lim_{\eta\to 0}\lim_{\epsilon\to 0}\lim_{R\to\infty}\overline{\lim_{j\to\infty}}\left(\sup_{S_{R+\sqrt{R},n}(\eta)} \mathbb{P}\left(\Omega_{x,n,R+\sqrt{R}}(\epsilon)\right) + \sup_{S_{R,n}(\eta)} \mathbb{P}\left(\Omega_{x,n,R}(\epsilon)\right) + \eta + \epsilon + R^{-1/2}\right) \\ &= \lim_{\eta\to 0}\lim_{\epsilon\to 0}\left(\lim_{R\to\infty}\overline{\lim_{j\to\infty}}\sup_{S_{R+\sqrt{R},n}(\eta)} \mathbb{P}\left(\Omega_{x,n,R+\sqrt{R}}(\epsilon)\right) + \lim_{R\to\infty}\overline{\lim_{j\to\infty}}\sup_{S_{R,n}(\eta)} \mathbb{P}\left(\Omega_{x,n,R}(\epsilon)\right)\right)\end{aligned}$$

which, upon application of Lemma 3.4, is zero.

By squeezing, we conclude Theorem 1.1:

$$\mathbb{E}\left|\frac{N(f_{n_j})}{n_j} - a_\nu\right| \to 0 \qquad (j \to \infty).$$

2. Exponential concentration

The method of proof for Theorem 1.2 consists in adapting Nazarov and Sodin's [**17**] for BARWs. This turns out to require less additional work than was needed to prove Theorem 1.1, which is perhaps not surprising, relying for the most part upon properties of high-dimensional Gaussian probability spaces, and then combining this with the conclusion of Theorem 1.1.

2.1. Proof of Theorem 1.2.

2.1.1. *Outline.* We begin with a preliminary reduction to proving concentration about the sequence of median values of $N(f_n)/n$, then introduce Lemma 4.1 and show that it implies concentration about the sequences of medians.

Remark. Large non-specific constants will usually be indicated by C, and small non-specific constants by c, and these may vary from line to line.

2.1.2. *Concentration about median implies concentration about limiting mean.* We now show that to prove exponential concentration about the limiting mean a_ν established in Theorem 1.1, it suffices to prove exponential concentration about the sequence of medians $m_n := \text{Median}(N(f_n)/n)$. Assume that for all $\epsilon > 0$ there exist positive constants $C(\epsilon)$, $c(\epsilon)$ such that

$$\mathbb{P}\left\{\left|\frac{N(f_n)}{n} - m_n\right| > \tfrac{1}{4}\epsilon\right\} \leq C(\epsilon)e^{-c(\epsilon)\mathcal{N}_n}. \tag{4.20}$$

Then, writing $\nu_n := \mathbb{E}(N(f_n)/n)$ for the sequence of means, we have

$$\begin{aligned}\mathbb{P}\left\{\left|\frac{N(f_n)}{n} - \nu_n\right| > \tfrac{3}{4}\epsilon\right\} &= \mathbb{P}\left\{\left|\frac{N(f_n)}{n} - m_n + m_n - \nu_n\right| > \tfrac{3}{4}\epsilon\right\} \\ &\leq \mathbb{P}\left\{\left|\frac{N(f_n)}{n} - m_n\right| + |m_n - \nu_n| > \tfrac{3}{4}\epsilon\right\}.\end{aligned} \tag{4.21}$$

Note that

$$|\nu_n - m_n| = \left|\mathbb{E}\left\{\frac{N(f_n)}{n} - m_n\right\}\right| \leq \mathbb{E}\left|\frac{N(f_n)}{n} - m_n\right|,$$

and also that Courant's nodal domain theorem (1.7) implies that a.s. the random variable $\frac{N(f_n)}{n}$ is uniformly bounded w.r.t. n by some constant ≤ 1, so that

$$\mathbb{E}\left|\frac{N(f_n)}{n} - m_n\right| \leq \mathbb{P}\left\{\left|\frac{N(f_n)}{n} - m_n\right| < \tfrac{1}{4}\epsilon\right\} \cdot \tfrac{1}{4}\epsilon + \mathbb{P}\left\{\left|\frac{N(f_n)}{n} - m_n\right| > \tfrac{1}{4}\epsilon\right\} \cdot 1.$$

Now using (4.20) gives

$$|\nu_n - m_n| \leq \tfrac{1}{4}\epsilon + C(\epsilon)e^{-c(\epsilon)\mathcal{N}_n} \leq \tfrac{1}{2}\epsilon,$$

provided $\mathcal{N}_n$ is taken sufficiently large; using this with (4.20) in (4.21) we get

$$\mathbb{P}\left\{\left|\frac{N(f_n)}{n}-\nu_n\right|>\tfrac{3}{4}\epsilon\right\}\leq\mathbb{P}\left\{\left|\frac{N(f_n)}{n}-m_n\right|>\tfrac{1}{4}\epsilon\right\}\leq C(\epsilon)e^{-c(\epsilon)\mathcal{N}_n},$$

and we have exponential concentration about the sequence of mean values (ν_n), provided that the sequence of values of n is taken so that $\mathcal{N}_n\to\infty$. Finally, to obtain concentration about the limiting mean a_ν, choose a sequence $(n_j)_{j\in\mathbb{N}}$ so that the hypotheses in Theorem 1.1 are met. We then have $\nu_{n_j}\to a_\nu$, so that for all j large enough, $\left|\nu_{n_j}-a_\nu\right|\leq\frac{1}{4}\epsilon$ and we conclude that

$$\mathbb{P}\left\{\left|\frac{N(f_{n_j})}{n_j}-a_\nu\right|>\epsilon\right\}\leq\mathbb{P}\left\{\left|\frac{N(f_{n_j})}{n_j}-\nu_{n_j}\right|+\left|\nu_{n_j}-a_\nu\right|>\epsilon\right\}\leq C(\epsilon)e^{-c(\epsilon)\mathcal{N}_{n_j}},$$

as claimed.

2.1.3. *An unstable nodal count is exponentially rare.* The key step in proving exponential concentration is demonstrating that for all but an exponentially small set of 'unstable' $f\in\mathcal{H}_n$, the nodal count $N(f)$ changes little when perturbed by a small-norm, but otherwise arbitrary, $g\in\mathcal{H}_n$. Lemma 4.1 makes this precise and is the BARW version of [**17**, Lemma 4]. The proof below is an adaptation of theirs and is structured very similarly. The interesting differences arise from the degeneracies of our ensemble, which become prominent between equalities (4.27) and (4.29).

LEMMA 4.1. *For all $\epsilon>0$, there exists $\rho>0$ and an exceptional set $E\subset\mathcal{H}_n$ of probability $\mathbb{P}(E)\leq C(\epsilon)e^{-c(\epsilon)\mathcal{N}_n}$ such that for all $f\in\mathcal{H}_n\setminus E$ and for all $g\in\mathcal{H}_n$ with $\|g\|\leq\rho$, we have*

$$N(f+g)\geq N(f)-\epsilon n.$$

We defer proof of Lemma 4.1 to Section 2.2, and proceed to show that together with the preliminary reduction it implies Theorem 1.2.

2.1.4. *Proof that Lemma 4.1 implies Theorem 1.2.* Recall the notation m_n for the median of the random variable $N(f_n)/n$, and consider the sets

$$F=\{f\in\mathcal{H}_n:N(f)>(m_n+\epsilon)n\}$$
$$G=\{f\in\mathcal{H}_n:N(f)<(m_n-\epsilon)n\}.$$

For the set F: if $f\in F\setminus E$, then a perturbation g of norm ρ cannot cause the count of nodal components to decrease by as many as ϵn, by Lemma 4.1. Therefore we have more than the median number of components, so that $\mathbb{P}\left((F\setminus E)_{+\rho}\right)\leq\frac{1}{2}$,

and so $\mathbb{P}(F \setminus E) \leq 2e^{-c\rho^2\mathcal{N}_n}$ by concentration of Gaussian measure (Lemma 2.8). Accounting for the exceptional set E, we see that

$$\mathbb{P}(F) \leq 2e^{-c\rho^2\mathcal{N}_n} + C(\epsilon)e^{-c(\epsilon)\mathcal{N}_n} \leq C(\epsilon)e^{-c(\epsilon)\mathcal{N}_n}. \tag{4.22}$$

For the set G: we first note that any $h \in G_{+\rho}$ may be written as $h = f + g$ for $f \in G$ and $\|g\| \leq \rho$. Moreover, if $h \in G_{+\rho} \setminus E$, then the perturbation $h - g = f$ satisfies $N(h-g) \geq N(h) - \epsilon n$, which, rewritten, reads $N(f+g) \leq N(f) + \epsilon n < m_n$. Therefore

$$G_{+\rho} \subset (G_{+\rho} \setminus E) \cup E \subset \{f \in \mathcal{H}_n : N(f) < m_n\} \cup E,$$
$$\mathbb{P}(G_{+\rho}) \leq \mathbb{P}\{f \in \mathcal{H}_n : N(f) < m_n\} + \mathbb{P}(E) \leq \tfrac{1}{2} + C(\epsilon)e^{-c(\epsilon)\mathcal{N}_n},$$

and so $\mathbb{P}(G_{+\rho}) \leq \frac{3}{4}$ for sufficiently large $\mathcal{N}_n$. Finally, using concentration of measure once more,

$$\mathbb{P}(G) \leq 2e^{-c\rho^2\mathcal{N}_n} \leq C(\epsilon)e^{-c(\epsilon)\mathcal{N}_n}. \tag{4.23}$$

Choosing the constants $C(\epsilon)$ to be sufficiently large, this holds for small $\mathcal{N}_n$ also. Together, (4.22) and (4.23) imply (4.20), and thus also Theorem 1.2.

2.2. Proof of Lemma 4.1.

2.2.1. *Outline.* Cover $\mathbb{S}$ with discs (of a radius to be specified below). We will see that, if $|f(x)|$ and $|\nabla f(x)|$ are never simultaneously small in a disc, then we can guarantee that its nodal components do not merge or disappear when perturbed by a small-normed g. Discs on which we can guarantee this will be called 'stable' for the f. Other discs will be called 'unstable' for f. If there aren't too many unstable discs for f, then the total nodal count $N(f+g)$ cannot fall significantly less than $N(f)$, which is what is needed in order to conclude Lemma 4.1. The additional subtlety in applying this method for the ensemble of BARWs over that of ARWs and RSHs, arises between the lines (4.27) and (4.29).

Remark. In the following section we will find ourselves making regular reference to functions $f \in \mathcal{H}_n$, and it will be convenient to refer to them as *samples*.

The remainder of this section consists in making the outline above precise. We split the argument into four parts:

(i) Definition of unstable samples
(ii) Unstable samples are exponentially rare
(iii) Stable samples deserve their name
(iv) Asserting valid parameters for the constraints derived in (ii) and (iii)

2.2.2. *Definition of unstable samples.* Let R be a large parameter. Fix a cover (with bounded covering multiplicity) of the unit square $\mathbb{S}$ by approximately n/R^2 discs $\mathcal{D}_j$ of radius $R/\sqrt{n}$. Now augment the cover: let $3\mathcal{D}_j$ denote the discs of radius $3R/\sqrt{n}$ with the same centres as the discs $\mathcal{D}_j$. Inevitably some will intersect $\partial\mathbb{S}$, but there will be at most $\ll \sqrt{n}/R$ such discs which are of negligible contribution. Fix small parameters $\alpha, \beta > 0$. We now "test the stability" of the samples $f \in \mathcal{H}_n$ against each disc of the augmented cover. A disc $3\mathcal{D}_j$ is called *stable* for f if there is no point $x \in 3\mathcal{D}_j$ for which $|f(x)| < \alpha$ and $|\nabla f(x)| < \beta$. Otherwise, $3\mathcal{D}_j$ is called *unstable* for f. Fix another small parameter $\delta > 0$. We call a sample f *unstable* if the number of unstable discs for f exceeds δn. Otherwise, we call f *stable*. We write E for the set of all unstable samples.[2] This will comprise the 'exceptional set' in Lemma 4.1.

2.2.3. *Unstable samples are exponentially rare.* Here, we deduce constraints on the parameters α, β, δ which will retain sufficient flexibility to guarantee that $E \subset \mathcal{H}_n$ is of exponentially decaying probability. First, note that $\mathbb{P}\{f \in \mathcal{H}_n : \|f\| > 2\} \leq e^{-c\mathcal{N}_n}$ by Corollary 2.7 (Bernstein's inequality), so that we may assume that $\|f\| \leq 2$. Now, suppose that $f \in E$. Then at least δn discs in the augmented cover are unstable for f. Let J be the set of indices of a sub-family of these δn unstable discs, any two discs of which have all of their points separated by at least $\frac{4}{\sqrt{n}}$. Since $4/\sqrt{n}$ is small compared with $R/\sqrt{n}$, there is such a collection with $|J| \geq c_1\delta n$. For each disc $3\mathcal{D}_j$ with $j \in J$, fix a point $x_j \in 3D_j$ at which $|f(x_j)| < \alpha$ and $|\nabla f(x_j)| < \beta\sqrt{n}$. Fix a parameter $\gamma \in (0,1)$ and let $M_j := \max_{x \in D(x_j, \gamma/\sqrt{n})} |\nabla\nabla f(x)|$. By construction, all members of the family of discs $\{D(x_j, 2/\sqrt{n})\}_{j\in J}$ are mutually disjoint, since the centres x_j must be separated by at least $4/\sqrt{n}$.

Now, we sum up the local analytic inequalities (2.5) applied to the family of disjoint discs $\{D(x_j, \gamma/\sqrt{n})\}_{j\in J}$, and play it off against the condition $\|f\| \leq 2$, to deduce an upper bound for the mean value of M_j^2. Explicitly:

$$M_j^2 = \max_{D(x_j,\gamma/\sqrt{n})} |\nabla\nabla f(x)|^2 \leq Cn^3 \int_{D(x_j,2/\sqrt{n})} |f(x)|^2\, dx,$$

$$\sum_{j\in J} M_j^2 \leq Cn^3 \sum_{j\in J} \int_{D(x_j,2/\sqrt{n})} |f(x)|^2\, dx \leq Cn^3\|f\|^2 \leq Cn^3,$$

[2] E depends on α, β, δ, R and n.

and so

$$\frac{1}{|J|}\sum_{j\in J} M_j^2 \leq \frac{Cn^3}{c_1\delta n} = C\delta^{-1}n^2.$$

From here

$$\#\{j \in J : M_j \geq 2\sqrt{C}\delta^{-1/2}n\} \leq \tfrac{1}{4}\,|J|\,,$$

and we deduce that in at least $\frac{3}{4}$ of the radius $\gamma/\sqrt{n}$ discs indexed by J, we have an upper bound of $C\delta^{-1/2}n$ for the quantity M_j, for some constant C. At all points in each of these discs, we can use the quantities α, β and M_j to guarantee upper bounds for $|f|$ and $|\nabla f|$:

$$\begin{aligned}
|f(x)| &\leq |f(x_j)| + |x-x_j|\,|\nabla f(x_j)| + C_1\,|x-x_j|^2\,M_j \\
&\leq \alpha + \beta\gamma + C_2\delta^{-1/2}\gamma^2, \\
|\nabla f(x)| &\leq |\nabla f(x_j)| + C_1\,|x-x_j|\,M_j \\
&\leq (\beta + C_2\delta^{-1/2}\gamma)\sqrt{n}.
\end{aligned}$$

Now perturb f by some $g \in \mathcal{H}_n$ for which $\|g\| \leq \tau$. The same argument again, now using the inequalities (2.3) and (2.4) applied to g, give bounds

$$\begin{aligned}
&\max_{x\in\mathcal{D}(x_i,\gamma/\sqrt{n})} |g(x)| \leq C\tau\delta^{-1/2}, \qquad i \in J_1 \subset J \qquad |J_1| \geq \tfrac{3}{4}\,|J| \\
&\max_{x\in\mathcal{D}(x_j,\gamma/\sqrt{n})} |\nabla g(x)| \leq C\tau\delta^{-1/2}\sqrt{n}, \qquad j \in J_2 \subset J \qquad |J_2| \geq \tfrac{3}{4}\,|J|\,.
\end{aligned}$$

Combining all of these upper bounds together, we deduce that on at least $\frac{1}{4}\,|J|$ of the discs indexed by J we have

$$|f+g| \leq A, \tag{4.24}$$

$$|\nabla(f+g)| \leq B\sqrt{n}, \tag{4.25}$$

where

$$\begin{aligned}
A &:= \alpha + \beta\gamma + C_3\delta^{-1/2}(\gamma^2+\tau) \\
B &:= \beta + C_3\delta^{-1/2}(\gamma+\tau).
\end{aligned}$$

Taking the measure of the union of these $\frac{1}{4}\,|J| \asymp \delta n$ (disjoint) discs, we deduce a lower bound on the measure of points for which the function and the gradient are simultaneously small. Explicitly, if we define

$$\mathcal{A}(h) = \text{Area}\{x \in \mathbb{S} : |h(x)| \leq A, |\nabla h(x)| \leq B\sqrt{n}\},$$

then the construction culminating in (4.24) and (4.25) shows that

$$f + g \in \left\{ h \in \mathcal{H}_n : \mathcal{A}(h) \geq c_3 \delta \gamma^2 \right\}.$$

We wish to use the concentration of Gaussian measure, and in pursuit of this, demonstrating that

$$\mathbb{P}\left\{ \mathcal{A}(h) > c_3 \delta \gamma^2 \right\} \leq \tfrac{1}{2}, \tag{4.26}$$

would imply that unstable f are exponentially rare. A natural approach to bounding (4.26), is to try to bound $\mathbb{E}\left\{\mathcal{A}(h)\right\}$ above and apply Markov's inequality. Writing out this expectation and manipulating using Fubini's theorem:

$$\begin{aligned}
\mathbb{E}\{\mathcal{A}(h)\} &= \int_\Omega \text{Area}\left\{ x \in \mathbb{S} : |h(x)| \leq A, |\nabla h(x)| \leq B\sqrt{n} \right\} d\mathbb{P} \\
&= \int_\Omega \int_{\mathbb{S}} \mathbf{1}_{\left\{ |h(x)| \leq A, |\nabla h(x)| \leq B\sqrt{n} \right\}} dx d\mathbb{P} = \int_{\mathbb{S}} \int_\Omega \mathbf{1}_{\left\{ |h(x)| \leq A, |\nabla h(x)| \leq B\sqrt{n} \right\}} d\mathbb{P} dx \\
&= \int_{\mathbb{S}} \mathbb{P}\left\{ |h(x)| \leq A, |\nabla h(x)| \leq B\sqrt{n} \right\} dx. \qquad (4.27)
\end{aligned}$$

Here the peculiarities of the ensemble of BARWs arise. Because the random field f_n degenerates at points in $\mathbb{S}$, we need to be careful when estimating the integral (4.27). Introduce a parameter $\varsigma > 0$ and define

$$\mathbb{S}_\varsigma := \{ x \in \mathbb{S} : \det \Sigma_x < \varsigma n^2 \}, \tag{4.28}$$

where Σ_x is the covariance matrix of the Gaussian random vector $(h(x), \nabla h(x))$. In this notation, we have

$$\begin{aligned}
\mathbb{E}\{\mathcal{A}(h)\} &= \int_{\mathbb{S}} \mathbb{P}\{ |h(x)| \leq A, |\nabla h(x)| \leq B\sqrt{n} \} dx \\
&\leq \int_{\mathbb{S}_\varsigma} dx + \int_{\mathbb{S} \setminus \mathbb{S}_\varsigma} \mathbb{P}\{ |h(x)| \leq A, |\nabla h(x)| \leq B\sqrt{n} \} dx \\
&\leq \text{meas}(\mathbb{S}_\varsigma) + \int_{\mathbb{S} \setminus \mathbb{S}_\varsigma} \mathbb{P}\{ |h(x)| \leq A, |\nabla h(x)| \leq B\sqrt{n} \} dx.
\end{aligned}$$

For each $x \in \mathbb{S} \setminus \mathbb{S}_\varsigma$ the covariance matrix Σ_x is positive-definite, hence the inverse $(\Sigma_x)^{-1}$ exists and is also positive-definite. Since the random vector $(h(x), \nabla h(x))$ is

jointly Gaussian, we have

$$\begin{aligned}\int_{\mathbb{S}\setminus\mathbb{S}_\varsigma} &\mathbb{P}\{|h(x)|\le A, |\nabla h(x)|\le B\sqrt{n}\}dx\\ &\le \int_{\mathbb{S}\setminus\mathbb{S}_\varsigma}\int_{\mathbb{R}^3}\frac{\exp(-\frac{1}{2}z(\Sigma_x)^{-1}z^T)}{(2\pi)^{3/2}(\det\Sigma_x)^{1/2}}\mathbf{1}_{\{|z_1|\le A,|(z_2,z_3)|\le B\sqrt{n}\}}dzdx\\ &\le \int_{\mathbb{S}\setminus\mathbb{S}_\varsigma}\frac{1}{(2\pi)^{3/2}(\det\Sigma_x)^{1/2}}\int_{\mathbb{R}^3}\mathbf{1}_{\{|z_1|\le A,|(z_2,z_3)|\le B\sqrt{n}\}}dzdx\\ &\le \int_{\mathbb{S}\setminus\mathbb{S}_\varsigma}\frac{1}{(2\pi)^{3/2}(\det\Sigma_x)^{1/2}}(2A)\pi(B\sqrt{n})^2dx\\ &\le \frac{(2A)\pi(B\sqrt{n})^2}{(2\pi)^{3/2}(\varsigma n^2)^{1/2}}\le \varsigma^{-1/2}AB^2\end{aligned}$$

and altogether

$$\mathbb{E}\{\mathcal{A}(h)\}\le \mathrm{meas}(\mathbb{S}_\varsigma)+\varsigma^{-1/2}AB^2.$$

But we already know how to estimate $\mathrm{meas}(\mathbb{S}_\varsigma)$: we can apply Chebyshev's inequality to the *unscaled* version of (3.11) with $u=0$. The only difference between the statement for the scaled and the unscaled matrix is in the constant, since partial differentiation in the scaled matrix accumulates an additional factor of $2^4/n^2$ (on account of the scaling factor in the variable). Multiplying through by $\frac{1}{16}n^2$, we have the immediate corollary:

COROLLARY 4.2. *For any fixed* $\varsigma<\frac{\pi^4}{4}$, *we have* $\mathrm{meas}(\mathbb{S}_\varsigma)\ll\mathcal{N}_n^{-1}$ *as* $\mathcal{N}_n\to\infty$.

Fixing such a ς we may assume that $\mathcal{N}_n$ is sufficiently large so that

$$\mathrm{meas}(\mathbb{S}_\varsigma)<\tfrac{1}{4}AB^2,$$

and provided that $\varsigma>16$ say, we have that

$$\mathbb{E}\{\mathcal{A}(h)\}<\tfrac{1}{4}AB^2+\varsigma^{-1/2}AB^2<\tfrac{1}{2}AB^2. \tag{4.29}$$

Finally, by applying Markov's inequality in the form

$$\mathbb{P}\{\mathcal{A}(h)\ge c_3\delta\gamma^2\}\le\frac{\mathbb{E}\{\mathcal{A}(h)\}}{c_3\delta\gamma^2},$$

and comparing with our desired inequality (4.26), we derive the condition

$$2AB^2\le c_3\delta\gamma^2, \tag{4.30}$$

which suffices to demonstrate an exponentially small set. The condition (4.30) can be met for fixed $\delta>0$, because the only parameter (other than δ) which features on both sides is γ, which is free to be chosen within the interval $(0,1)$. Because the

l.h.s. behaves like γ^4, while the r.h.s. behaves like γ^2, we can always choose small positive γ satisfactorily.

2.2.4. *Stable samples deserve their name.* Having shown that the samples we called 'unstable' are exponentially rare, we now need to justify our definition of 'stable' samples: those with fewer than δn unstable discs. The key tool in showing this is the 'shell lemma', Lemma 2.14, which we discussed in Chapter 2. If f and ∇f are never simultaneously small in a disc $\mathcal{D}$, and if f is perturbed by a small-norm sample g, then away from the boundary $\partial\mathcal{D}$, distinct nodal components of f are in correspondence with distinct nodal components of $f+g$. Therefore the nodal count in a stable disc $\mathcal{D}$ cannot suddenly drop (cf. Lemma 4.1).

Stable samples may also however have unstable discs. We begin by justifying that those unstable discs are not numerous enough to affect exponential concentration. Firstly, Lemma 2.14 will be applied to the discs $3D_j$, of radius $3R/\sqrt{n}$: any nodal components of diameter greater than $R/\sqrt{n}$ cannot be handled. But the Donnelly–Fefferman length estimate (Lemma 2.4) gives $\mathcal{L}(Z(f_n)) \ll \sqrt{n}$, so that

$$N_{R-long}(f) \ll \tfrac{\sqrt{n}}{R/\sqrt{n}} = \tfrac{n}{R} \ll \epsilon n,$$

provided that the condition $1/R \ll \epsilon$ holds. Secondly, Lemma 2.14 cannot be applied to unstable discs: nodal components contained within unstable discs cannot be handled. But by the Faber–Krahn estimate (Lemma 2.2) there are $\ll n \cdot \delta(R/\sqrt{n})^2$ such components. Provided that the condition $\delta R^2 \ll \epsilon$ holds, these are of negligible contribution.

For fixed, stable $f \in \mathcal{H}_n$, and an arbitrary perturbation $g \in \mathcal{H}_n : \|g\| \leq \rho$, we now apply Lemma 2.14. Let $\mathcal{D} = 3D_j$ be a stable disc for f, let $\mu = \alpha$ and $\nu = \beta\sqrt{n}$. Suppose that a component $\gamma \in Z(f)$ has $\operatorname{diam}(\gamma) \leq R/\sqrt{n}$ and intersects the disc D_j. Then $\operatorname{dist}(\gamma, \partial\mathcal{D}) \geq R/\sqrt{n} \geq \alpha/(\beta\sqrt{n})$, provided the condition $\alpha/\beta \leq R$ is satisfied. The final hypothesis is that $\sup_{3D_j} |g| < \alpha$. Suppose now that J indexes a family of disjoint discs for which this doesn't hold. Then by the local estimate (2.3) for the discs indexed by J, we have

$$|J|\,\alpha^2 \leq \sum_{j\in J} \left(\sup_{x\in\mathcal{D}} |g(x)|^2 \right) \ll n \int_{\mathcal{D}_{+1/\sqrt{n}}} |g(x)|^2\, dx \ll n\rho^2.$$

By the Faber–Krahn estimate, each $3D_j$ contains $\ll R^2$ nodal components and so we can ignore the discs with $\sup_{3D_j} |g| \geq \alpha$, provided that we ensure the condition that $\rho^2\alpha^{-2}R^2 \ll \epsilon$.

The remaining discs $\{3D_j\}_{j\notin J}$ satisfy the hypotheses of Lemma 2.14, and for the corresponding small discs D_j in the cover of $\mathbb{S}$, the nodal count does not decrease upon perturbation.

2.2.5. *Choosing parameters.* From the preceding sections we have accumulated the parameters $\alpha, \beta, \gamma, \delta, \tau, \rho, R$, subject to the derived conditions:

$$R \gg \tfrac{1}{\epsilon}, \qquad \delta R^2 \ll \epsilon, \qquad \alpha \ll R\beta, \qquad \rho^2\alpha^{-2}R^2 \ll \epsilon, \tag{4.31}$$

and

$$\left(\alpha + \beta\gamma + C_3\delta^{-1/2}(\gamma^2+\tau)\right)\left(\beta + C_3\delta^{-1/2}(\gamma+\tau)\right)^2 \ll \delta\gamma^2. \tag{4.32}$$

We have noted already that (4.32) can be satisfied for any fixed positive α, β, δ. The conditions (4.31) can be met, as, considering the conditions from left to right, there is always at least one new free parameter to specify. Moreover, the conditions derived are precisely the same as those in [**17**], and [**21**], thus we can attain $c(\epsilon) \asymp \min(\rho^2, \tau^2) \asymp \epsilon^{15}$ as the constant of the exponent in Theorem 1.2 with a valid choice of the parameters.

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